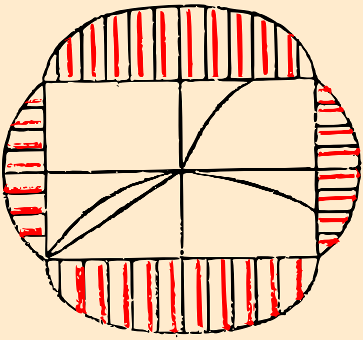
Kinasoshvili

STRENGTH OF MATERIALS



Р. С. КИНАСОШВИЛИ

СОПРОТИВЛЕНИЕ МАТЕРИАЛОВ

ИЗДАТЕЛЬСТВО «НАУКА»
МОСКВА

Strength of Materials

R. KINASOSHVILI

Translated from the Russian

by

M. Konyaeva

MIR PUBLISHERS MOSCOW

First published 1972
Second printing 1978

TO THE READER

Mir Publishers welcome your comments on the contents, translation and design of the book.

We would also be pleased to receive any proposals you care to make about our future publications.

Our address is:

Mir Publishers, 2 Pervy Rizhsky Pereulok, Moscow, USSR.

Printed in the Union of Soviet Socialist Republics

На английском языке

Contents

Chapter XIII.

Chapter XIV.

Notation

A — area
$A, B, C, \ldots$ — points, reactions at supports
$a, b, c, \ldots$ — distances
a_k — modulus of toughness
b — width
C — centroid or centre of gravity, constant of integration
D — constant of integration, diameter
d — diameter, differential of (as dx)
E — modulus of elasticity in tension or compression
EI — flexural rigidity
e — eccentricity, exponential constant (2.71828...)
f — deflection
G — modulus of elasticity in shear
GI_p — torsional rigidity or stiffness
g — acceleration of gravity
H — height, horizontal component of reaction, horsepower
h — depth of beam, height
hp — horsepower
I — moment of inertia
I_p — polar moment of inertia
I_x, I_y, I_z — moment of inertia with respect to x, y and z axes
$I_{xy}, \ldots$ — product of inertia
i — radius of gyration
k — factor of safety, impact factor, stability factor
k_σ — factor of safety based on normal stresses
k_τ — factor of safety based on shearing stresses
kgf — kilogram-force
l — distance, length
l_{eff} — effective length
M — bending moment
M_{lim} — limiting moment
M_t — twisting moment (torque)
m — mass, moment
N — normal force, power
n — revolutions per minute
O — origin of co-ordinates

P—concentrated load, force
P_{al}—allowable load
P_{cr}—critical force (Euler's)
P_{lim}—limiting load
p—pressure, total stress
Q—shearing force
q—load intensity (load per unit length)
R—radius, reaction, resultant force
r—radius, stress ratio
rpm—revolutions per minute
S—static moment
s—arc length
T—tension, torque
t—temperature
U—strain energy
u—strain energy per unit volume
V—vertical force
v—velocity, volume
W—weight, work
X, Y, Z—unknown forces
x, y, z—rectangular co-ordinates
x_c, y_c, z_c—rectangular co-ordinates of centroid or centre of gravity
y—deflection
Z—section modulus
Z_p—polar section modulus
α (alpha)—angle, coefficient of linear expansion, numerical factor, stress concentration factor
α_{eff}—effective stress concentration factor
β (beta)—numerical factor
γ (gamma)—angle of shear or shearing strain, numerical factor, specific weight (weight per unit volume)
γ_{xy}, γ_{yz}, γ_{zx}—shearing strains in xy, yz and zx planes
Δl (delta)—total elongation
δ (delta)—percentage elongation at rupture, thickness
ε (epsilon)—normal strain, size effect factor
ε_x, ε_y, ε_z—strains in x, y and z directions
ε_1, ε_2, ε_3—principal strains
θ (theta)—angle of twist per unit length
λ (lambda)—slenderness ratio
μ (mu)—length reduction factor, Poisson's ratio
π (pi)—3.14159...
ρ (rho)—radial distance, radius of curvature
σ (sigma)—normal stress
$[\sigma]$—allowable stress

σ_a stress amplitude
σ_b—bearing stress
σ_{cr}—critical stress
σ_e—elastic limit
σ_{eq}—equivalent stress
σ_m—mean stress
σ_p—proportional limit
σ_r—endurance limit
σ_u—ultimate stress
$\sigma_{u.c}$, $\sigma_{u.t}$—ultimate stresses in compression and tension
σ_x, σ_y, σ_z—normal stresses on planes perpendicular to x, y and z axes
σ_y—yield point stress
σ_1, σ_2, σ_3—principal stresses
σ_{-1}—endurance limit for completely reversed cycle
τ (tau)—shearing stress
$[\tau]$—allowable stress in shear
τ_u—ultimate shearing stress
τ_y—yield point stress in shear
τ_{-1}—endurance limit in shear
φ (phi)—allowable stress reduction factor, angle, slope of beam, total angle of twist
ψ (psi)—angle, percentage reduction in area
ω (omega)—angular velocity

Chapter I

Introduction

1. Science of Strength of Materials. Concepts of Deformation and of an Elastic Body

During operation structures and machines are acted on by external loads; for instance, the abutments of a railway bridge carry the weight of a moving train and the weight of the bridge itself, the connecting rod of an automobile engine is subjected to gas pressure in the cylinder. In order that structural elements and machine parts may sustain the loads acting on them without fracture or appreciable deformation, they must be made of a proper material and have the necessary dimensions. These dimensions of structural members are determined by calculation.

The development of the foundations for the design of structural members is the subject matter of a science called the strength of materials.

The dimensions of a member are determined with due allowance for the properties of the material of which the member is to be made. To choose the material rationally and to utilize it most efficiently one must have information regarding the most important properties of various structural materials (steel, cast iron, wood, concrete, stone, etc.). These data include in the first place the *strength* of a material, i. e., its ability to resist external loads without fracture.

Experimental studies of the strength of materials receive widespread attention at the present time. The science of strength of materials is, on the one hand, related to the science of materials and materials testing and, on the other hand, closely related to theoretical mechanics. Strength of materials is based on the laws and theorems of theoretical mechanics and uses its principles as long as they are consistent with the fundamental principles and the problems of strength of materials. For the solution of these problems a number of new concepts are introduced in strength of materials.

The most important of these are the concepts of *strain* and *stress*. In theoretical mechanics, solids are conventionally considered as absolutely rigid bodies, i. e., as bodies undergoing no change in shape under the action of forces applied to them. Experimental observations show, however, that all solids deform when subjected to forces.

The deformation of solids under the action of external forces is one of their basic properties. Besides, solids are capable of resisting changes in the relative position of their particles. This gives rise to forces inside the body, which resist its deformation and tend to restore the particles to the positions they occupied before the deformation. These forces are called *internal forces* or *elastic forces*, and the property of solids to "liquidate" the deformation due to external forces after they cease to act is called *elasticity*. A measure for the evaluation of internal elastic forces is *stress* (intensity of internal forces; for more details see Sec. 4).

Perfectly elastic or absolutely elastic bodies are those in which the deformation produced by external forces disappears completely after they cease to act. Perfectly inelastic bodies are those which completely retain, after the external forces cease to act, the deformation produced in them. There are neither perfectly elastic nor perfectly inelastic bodies in nature. However, such materials as steel, wood, etc., may be closely approximated by perfectly elastic bodies. But even these materials can be regarded as perfectly elastic only up to certain limits of loading established by experiment. Beyond these limits, after the external forces are removed there remains a deformation in the body which cannot be neglected.

Deformation which disappears completely after the external forces cease to act is called *elastic deformation*. Deformation which does not disappear is called *permanent* or *plastic deformation*. In engineering design, structural parts are, as a rule, so dimensioned and proportioned as to avoid permanent deformations.

As stated above, external forces acting on a solid produce internal forces which resist the external ones. Thus, for instance, if external forces stretch a solid, the internal forces will resist the stretching; there will act forces of mutual attraction between individual particles of the solid. As external forces increase, so do internal forces. The internal forces in each material can increase only to a certain limit characteristic of this material. The external forces may be so large that the internal forces in a body of given dimensions will not be able to balance them and the body will fracture.

Now that we have become acquainted with the concepts of deformation and internal elastic forces, we can say more about the problems of strength of materials. Namely: for various types of external forces strength of materials establishes mathematical relations between external forces, geometric proportions of structural members, the resulting elastic forces and deformations. These relations and the strength characteristics of materials are used to determine the required dimensions of structural members. In establishing these relations certain assumptions and limitations are made. These assumptions and limitations are necessary because it is impossible to cover all the features of the phenomena under study as a whole.

First of all, the material of which the structures are made is considered to be continuous, homogeneous at all points of the body and having the same properties in all directions. The latter property of the material is called isotropy. Indeed, some structural materials, such as cast metal, possess high homogeneity (cast iron is an exception in this case). Other materials, such as wood, possess lower homogeneity as compared with metals. The more homogeneous the material and the more alike its properties in different directions, the closer is the agreement between theoretical and experimental results.

Strength of materials, as a rule, deals with only those problems of the behaviour of bodies under the action of external loads in which the deformation is small compared with the dimensions of the body. This makes it possible to neglect the changes (produced by the deformation) in the position of the forces acting on the body. In addition to the assumptions listed above, some other assumptions are made in strength of materials, which will be introduced in the appropriate sections of the book.

In choosing the material and determining the proportions and dimensions of structural members, the basic factors to be taken into account are: the conditions in which the member will operate, the requirements on its strength, lifetime and economy.

In some cases additional special requirements are imposed on structural parts to be designed; thus, in designing aircraft members and engines the special requirement is a minimum weight. Of course, different requirements are placed on temporary structures which are built, say, for the duration of an emergency and on structures which are to serve for many years. Some of the requirements placed on a structure are in contradiction with each other, such as strength, light weight and economy. Thus, increasing the wall thickness of a cylinder of an aircraft piston engine increases the strength and reliability of the cylinder, but its weight becomes heavier; also when the crankshaft of the same engine is drilled out because of weight requirements, the shaft becomes lighter, but the cost of machining and hence the total cost is increased. The contradiction of these requirements prompts the development of the science of strength of materials.

In choosing the material and dimensioning the parts of a structure it is necessary to take into account simultaneously all the requirements placed on the structure, both the basic (strength, lifetime, economy) and special ones.

Without knowledge of the fundamentals of strength of materials it is impossible to construct even a simple machine satisfying the technical requirements placed on each construction. In strength of materials experiment and theory are closely interrelated; this science is simultaneously a theoretical and experimental one. All theoreti-

cal assumptions and conclusions are verified in practice and only after their validity is confirmed are they accepted for use. Experiment also comes to the rescue when theory cannot solve a problem because of its extreme complexity.

With the development of engineering mechanics the science of strength of materials is gaining in importance.

The ancient builders, having no theory whatsoever, were guided only by crude experience, copying known models; their structures were heavy and were sometimes built for centuries. With the development in the XVII century of international naval trade, metallurgy, mining it became necessary to solve more complicated problems of the strength of ships and structures. The old methods could no longer be used. The science of strength of materials originated at that time, the first course being published only in 1826 in France.

The first investigations in the field of strength were made by Galileo Galilei in the first half of the XVII century. In 1678 Robert Hooke formulated, on the basis of some observations, an important law which states that the amount of deformation in an elastic body is proportional to the load.

2. Classification of External Forces

External forces (loads) may act on machine and structural parts in different ways. According to the manner in which they are applied forces may be divided into body and surface forces. Among body forces is, for example, the gravity force (weight). Surface forces are divided into distributed and concentrated ones. Distributed forces are applied over an area or along a length. Thus, a snow layer on a roof is a load distributed over an area; gas pressure on the walls of a vessel is also a distributed load. Such loads are expressed in units of force per unit area (tons/m^2, kgf/cm^2, i. e., in tons or kilograms-force per square metre or square centimetre).

A load distributed along a length is expressed in units of force per unit length (tons/m, kgf/cm). A load may be uniformly or non-uniformly distributed along a length or over an area. For instance, the water pressure on a dam is non-uniformly distributed: the pressure increases with increasing depth.

Concentrated forces act over a very small area. A concentrated force is considered to be applied at a point for the sake of simplicity; this simplification introduces no serious error in calculations, as a rule. Concentrated forces are measured in units of force, i. e., in kilograms-force, tons *.

* If the analysis of structures is carried out in the new system of units (SI), use should be made of the relationships: 1 kgf = 10 newtons (N); 1 ton = 10 kilonewtons (kN); 1 kgf/cm^2 = 10 N/cm^2 = 100 kN/m^2 = 1 atm. In accurate calculations 10 should be replaced by 9.81.

According to their nature of action loads are divided into static and dynamic ones. A static load is defined as a load which increases slowly from zero to a certain maximum value and then remains constant or varies only slightly.

An example of dynamic loads is an impact load (the action of the ram of a steam hammer on a pile) when the time duration of the load is a small fraction of a second. Dynamic loads also include periodic loads varying in time. An example of such loads is the load on the connecting rod of an engine which varies continually in magnitude and direction. The number of reversals of such a load during the operation of the rod amounts to many millions. Finally, dynamic loads include inertia forces developed during vibration.

The division of loads into static and dynamic ones depends upon the fact that materials resist these types of loads in different ways.

3. Basic Types of Deformation

Deformations of structural and machine elements produced by external forces may be very complex. However, these complex deformations can always be represented as consisting of a small number of basic types of deformation.

The basic types of deformation of structural members which are studied in strength of materials are: (1) tension (Fig. 1), (2) compression (Fig. 2), (3) shear (Fig. 3), (4) torsion (Fig. 4), and (5) bending (Fig. 5).

Examples of complex deformations are provided by combined tension and torsion (Fig. 6) or combined tension and bending (Fig. 7).

The above types of deformation will be considered in detail and methods for determining strains and stresses will be given in the relevant chapters of the book. It should be noted that strength of materials deals with only simple-shaped bodies. These are rods, plates and thin-walled shells.

A rod is a body whose length is considerably greater than the transverse dimensions which are of the same order of magnitude. The axis AB of a rod may be curved or straight (Fig. 8*a* and *b*). Rods with straight axes are called bars, beams, columns, depending on their purpose.

A plate and a thin-walled shell are bodies whose thickness is considerably smaller than the other two dimensions. For instance, boilers, tanks, various vessels are thin-walled shells; the flat bottom of a boiler is a plate. Strength of materials deals mainly with rods. In the sequel we shall consider rods with straight axes and almost invariably of uniform section.

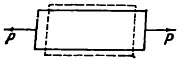


Fig. 1

Fig. 2

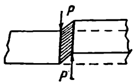


Fig. 3

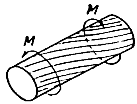


Fig. 4

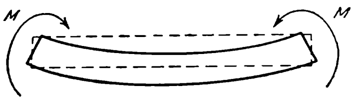


Fig. 5

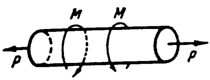


Fig. 6

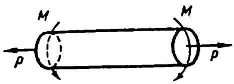


Fig. 7

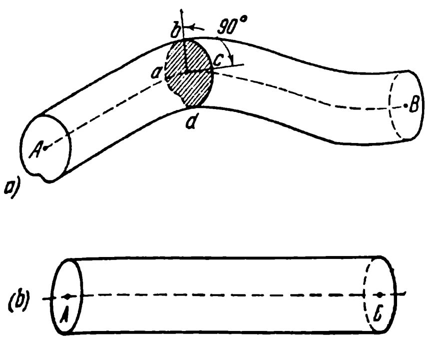


Fig. 8

In machine design elements of complex shape are sometimes encountered. Such elements cannot be handled by the methods of strength of materials. However, most machine parts can be treated approximately as rods using the methods of strength of materials. The results thus obtained may be refined by experiment.

At present, wide use is made in practice of experimental methods of strain measurement which make it possible to determine sufficiently accurately stresses in complex-shaped members which do not lend themselves to theoretical calculation. In the first place mention should be made of the application of wire resistance strain gauges which indicate stresses through the change of electrical resistance.

Problems involving the accurate determination of strains and stresses are dealt with in a science called the theory of elasticity. It uses rigorous mathematical methods. In practice, however, the design of machine and structural parts often does not require too much accuracy; it should be just sufficient but the methods of analysis should be simple and thus easy to apply. It is therefore customary in the design of machines and structures to use the methods of strength of materials which are considerably simpler than those of the theory of elasticity and give sufficiently accurate results. There are, however, problems which are solvable only by the methods of the theory of elasticity, such as the determination of stresses in balls or rollers of bearings. A simplification of the methods of analysis in strength of materials is achieved by introducing some assumptions.

Both the theory of elasticity and strength of materials usually consider elastic deformations. In engineering practice, however, there are many cases where a material develops plastic deformations. Plastic deformations are studied in a science called the theory of plasticity which has been extensively elaborated in the last few years.

4. Method of Sections. Stress

As stated above, external forces acting on a body give rise to internal resisting forces.

The external forces deform the body, the internal forces tend to retain its original shape and volume.

To solve problems of strength of materials it is necessary to know how to determine internal forces and deformations in a body. The internal forces at any section of a body are determined by the method of sections. The idea of this method is as follows.

Consider a body which is in a state of equilibrium under the action of forces P_1, P_2, P_3 and P_4 (Fig. 9*a*).

If, for instance, we are interested in the internal forces acting at a section *ab*, we imagine the body cut through this section and

one of the two parts removed, say, the right one. The remaining left-hand part (Fig. 9*b*) will then be acted on by the external forces P_1 and P_2. In order for this part of the body to remain in equilibrium, it is necessary to apply internal forces over the entire section.

These forces represent the action of the removed right-hand part of the body on the remaining left-hand part. Being internal forces for the entire body, they play the role of external forces for the isolated part. The magnitude of the resultant of the internal forces

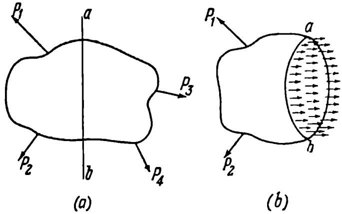


Fig. 9

can be determined from the condition of equilibrium of the isolated part. The law of distribution of internal forces over the section is not in general known. To solve this problem, it is necessary to know in each particular case how the body deforms under the action of external forces. Thus, the method of sections only allows us to determine the sum of the internal forces acting at the section in question. The sum of these forces may reduce to a single force, to a couple or, in the general case, to a force and a couple.

If an infinitesimal area ΔA is isolated at the section, it may be said, assuming the internal forces to be acting at all points in the section, that this area is acted on by an infinitesimal force ΔP. The ratio of the internal force ΔP to the magnitude of the isolated area ΔA gives the *average stress* on this area

$$p_{av}=\frac{\Delta P}{\Delta A}.$$

Thus, the stress (which characterizes the intensity of internal forces) is defined as the force per unit area. The stress is expressed in kilograms-force or newtons per square centimetre (kgf/cm^2, N/cm^2) or in kilograms-force or newtons per square millimetre (kgf/mm^2, N/mm^2). Reducing the area to zero, i.e., passing to the limit, we obtain the true stress at a given point, say, the centre of the area ΔA.

Consequently, the true stress at a given point is

$$p_{true} = \lim_{\Delta A \to 0} \frac{\Delta P}{\Delta A} = \frac{dP}{dA}. \tag{1.1}$$

If the internal forces (elastic forces) are known to be uniformly distributed over the section (Fig. 10), in this simplest case the stress is calculated by dividing the total elastic force acting at the section by the entire cross-sectional area, i. e.,

$$p = \frac{P}{A}. \tag{1.2}$$

Since the force has a direction, the stress will also have a direction.

In the general case, the stress (p) on a given area dA will make an angle α with this area (Fig. 11). Resolving this stress into two

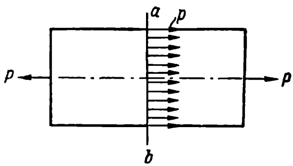


Fig. 10

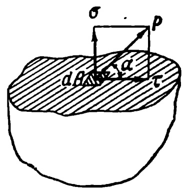


Fig. 11

components, one being perpendicular to the area, called the *normal stress* and designated by the letter σ (sigma), and the other lying in the plane of the area, called the *shearing* (or *tangential*) *stress* and designated by the letter τ (tau), we obtain

$$\sigma = p \sin \alpha, \quad \tau = p \cos \alpha.$$

The total stress is expressed in terms of the normal and shearing stresses by the following formula

$$p = \sqrt{\sigma^2 + \tau^2}. \tag{1.3}$$

The total stress is not considered to be a convenient measure of internal forces in a body as materials resist normal and shearing stresses in different ways. Normal stresses tend to bring closer together or separate individual particles of a body in the direction of the normal to the plane of the section. Shearing stresses tend to move particles of a body with respect to each other on the plane of the section.

In determining the stress at any point of a body, it is possible to pass an infinite number of differently oriented planes through this point. To fully characterize the state of stress at a given point, we have to know not only the magnitude and direction of the stress but also the inclination of the plane. In the following we shall see how the stress at a given point varies with the inclination of a plane passed through this point.

The concepts of strain and stress are the fundamental concepts in strength of materials.

5. Check Questions

What is deformation of a body?

What is elasticity of a body?

What is elastic deformation and plastic deformation?

What problems are dealt with in strength of materials?

What are the basic design requirements placed on machines and structures?

How are loads which act on machine and structural parts classified?

What is a rod, plate and thin-walled shell?

What are the basic types of deformation produced by external forces?

What is the idea of the method of sections?

What is stress?

What is the dimension of stress?

What is normal stress and shearing stress?

To familiarize yourself with the SI system of units, express 1 N, 1 kN in kgf and tons.

Express 1 N/cm^2 in kgf/cm^2 and 1 ton/m^2 in kN/m^2.

Chapter II

Tension and compression

6. Longitudinal Strain. Stress. Hooke's Law

Take a prismatic rod (Fig. 12) of constant cross-sectional area A cm^2. Mark two thin lines l mm apart on its surface using a sharp needle. Now apply two equal and opposite forces, each of P kgf, at the ends of the rod so that these forces will act precisely along the axis of the rod. The rod, being in equilibrium under the action of the tensile forces, will elongate in the longitudinal direction and its transverse dimensions will somewhat reduce.

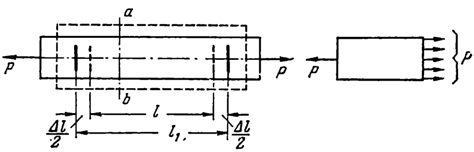


Fig. 12

We shall assume that all plane sections normal to the axis of the rod remain plane and normal to its axis after deformation. This hypothesis is known as the *hypothesis of plane sections*. It is supported by experimental evidence for sections sufficiently far removed from the point of application of the force P; by accepting this hypothesis it is assumed that all longitudinal elements of the rod are stretched in the same manner.

By measuring carefully the distance between the two lines marked on the surface, we find it increased and equal to l_1 mm. The elongation of the rod in the portion l is

$$\Delta l = l_1 - l.$$

This increment of the length of the rod is called the *total* or *absolute elongation in tension*; in the case of compression it is called the *total* or *absolute contraction*. In the latter case the quantity Δl has a negative sign.

The absolute elongation depends obviously on the original length of the rod. Therefore, a more convenient measure of defor-

mation is the elongation per unit of original length of the rod. The ratio

$$\varepsilon=\frac{\Delta l}{l}\,\frac{\text{absolute elongation, mm}}{\text{original length, mm}} \tag{2.1}$$

is termed the *longitudinal strain* or the *unit elongation*. The unit elongation has no dimension, it is a pure number and is often expressed as a percentage of the original length

$$\varepsilon\,\% = \frac{\Delta l}{l}\times 100 = \varepsilon\times 100\,\%.$$

To determine the stress at a transverse section, i.e., at a section perpendicular to the axis of the rod, we apply the general method accepted in strength of materials, viz. the method of sections.

Imagine the rod (Fig. 12) cut into two parts by a transverse section *ab* and the right-hand part removed. To hold the remaining left-hand part in equilibrium, we apply, in the plane of the section, internal elastic forces normal to the plane of the section. These forces replace the action exerted by the removed right-hand part on the left-hand part of the rod. The resultant elastic force will act along the axis of the rod and will be equal to P kgf. Accepting the hypothesis of plane sections, we thereby assume that in tension the elastic forces are uniformly distributed over the whole section; therefore, the stress at any point in the cross section is given by the formula

$$\sigma=\frac{P}{A}\ \text{kgf/cm}^2. \tag{2.2}$$

This stress will be normal since it acts, like the force P, perpendicular to the plane of the cross section. If the force P is measured in kilograms-force and the area A in square centimetres, then the stress σ will have the dimension kgf/cm^2.

In the case of compression the stress is calculated by the same formula (2.2), since only the direction of the forces is reversed here.

The magnitude of the stress in tension or compression is independent of the choice of section along the length of the rod. At any cross section the distribution of elastic forces is assumed to be uniform, and only at sections near the point of application of the external force uniform stress distribution is not to be expected. The determination of stresses at such locations is a difficult problem which is beyond the scope of a course in strength of materials.

The loads and the deformations produced in a rod are closely related. This relationship between load and deformation was first formulated by Robert Hooke in 1678. According to Hooke's law deformation is proportional to load. This is one of the fundamental laws in strength of materials. For a rod in tension or comp-

ression, Hooke's law expresses direct proportionality between stress and strain

$$\sigma = E\varepsilon. \tag{2.3}$$

This proportionality is violated when the stress exceeds a certain limit called the *proportional limit*. The proportional limit for materials is established by experiment.

The factor E appearing in formula (2.3) is known as the *modulus of elasticity* of the first kind or *Young's modulus*, after the name of the physicist who introduced it into the science. From formula (2.3) it is seen that the dimension of the modulus of elasticity E is the same as that of stress since ε is a dimensionless quantity, i.e., E is expressed in kgf/cm². For one and the same stress, the strain will be smaller for a material for which E is larger. Consequently the modulus of elasticity characterizes the stiffness of the material, i.e., its ability to resist deformation, a fact which follows from formula (2.3)

$$\varepsilon = \frac{\sigma}{E}.$$

The magnitude of the modulus of elasticity of materials is established experimentally. In Table 1 are given average values of E for some materials at room temperature.

Table 1. Moduli of Elasticity

Material	E, kgf/cm²
Steel	2×10^6 to 2.2×10^6
Cast iron	0.75×10^6 to 1.6×10^6
Copper	1×10^6
Bronze	1.2×10^6
Titanium	1.0×10^6
Aluminium	0.675×10^6
Wood	1×10^5

For materials which do not obey Hooke's law, such as stone, cement, leather, cast iron, etc., a power relation is used: $\sigma^m = E\varepsilon$. The exponent m, which is sometimes close to unity, is chosen experimentally.

Formula (2.3), which expresses Hooke's law, may be written in an alternate form substituting the appropriate expressions for σ and ε

$$\sigma = \frac{P}{A} \quad \text{and} \quad \varepsilon = \frac{\Delta l}{l};$$

we then obtain

$$\Delta l = \frac{Pl}{EA} \cdot \tag{2.4}$$

From this formula it follows that the elongation (contraction) of the rod is directly proportional to the tensile (compressive) force and the length of the rod, and inversely proportional to the cross-sectional area and the modulus of elasticity of the material. Sometimes the moduli in compression and tension are not equal (cast iron).

The product in the denominator of formula (2.4), i.e., EA is termed the *stiffness* in tension (compression). The greater the stiffness of the rod, the smaller is the deformation for one and the same length of the rod. The stiffness characterizes simultaneously the physical properties of the material and the geometric dimensions of the section. Formula (2.2) for stress and Hooke's law (2.3) or (2.4) are fundamental formulas in design for tension and compression.

Example 1. Determine the unit elongation of a bar if its original length is $l = 250$ mm and its length after extension $l_1 = 250.5$ mm.

Solution. The absolute elongation of the bar is

$$\Delta l = l_1 - l = 250.5 - 250 = 0.5 \text{ mm}.$$

The unit elongation of the bar is

$$\varepsilon = \frac{\Delta l}{l} = \frac{0.5}{250} = 0.002.$$

Expressing the unit elongation in per cent, we obtain

$$\varepsilon\% = 0.002 \times 100 = 0.2\%.$$

Example 2. A round bar of diameter $d = 2$ cm and length $l = 2$ m undergoes an absolute elongation $\Delta l = 0.5$ mm under the action of a tensile force $P = 800$ kgf. Determine the modulus of elasticity E of the material if the stress in the bar is not to exceed the proportional limit.

Solution. From formula (2.4) we have

$$E = \frac{Pl}{A\,\Delta l} = \frac{800 \times 200}{\frac{3.14 \times 2^2}{4} \times 0.05} = 1{,}020{,}000 \text{ kgf/cm}^2.$$

Example 3. Determine the stress, unit and absolute elongation (neglecting the effect of gravity) in a steel rod if the tensile force is $P = 3$ tons, the length of the rod $l = 2$ m, the cross-sectional area $A = 4$ cm^2. The proportional limit of the steel is 2,500 kgf/cm^2, the modulus of elasticity $E = 2 \times 10^6$ kgf/cm^2.

Solution. The stress in the rod is, by formula (2.2),

$$\sigma = \frac{P}{A} = \frac{3{,}000}{4} = 750 \text{ kgf/cm}^2.$$

Since the stress obtained is less than the proportional limit ($750 < 2{,}500$), the strain is proportional to the stress. Using formula (2.3), we have

$$\varepsilon = \frac{\sigma}{E} = \frac{750}{2 \times 10^6} = 0.000375.$$

The absolute elongation of the rod is

$$\Delta l = \varepsilon l = 0.000375 \times 200 = 0.075 \text{ cm}.$$

Example 4. A steel bolt 160 mm long undergoes an elongation $\Delta l = 0.12$ mm during tightening. The modulus of elasticity of the material is $E = 2 \times 10^6$ kgf/cm^2. Determine the stress in the bolt.

Solution. The unit elongation is

$$\varepsilon = \frac{\Delta l}{l} = \frac{0.12}{160} = 0.00075.$$

The stress in the bolt is determined from formula (2.3)

$$\sigma = E\varepsilon = 2 \times 10^6 \times 0.00075 = 1{,}500 \text{ kgf/cm}^2.$$

7. Lateral Strain in Tension and Compression

Experiments show that even if a rod undergoes very small deformations in the longitudinal direction its lateral dimensions change. An elongation in the longitudinal direction produces a contraction in the transverse direction, and conversely the shortening in the longitudinal direction is accompanied by a lateral expansion. Consequently, a body under tension lengthens and becomes thinner (Fig. 13), and under compression it shortens and becomes thicker. Lateral strains in tension or compression are proportional to longitudinal strains.

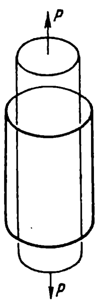


Fig. 13

If the longitudinal strain is denoted by ε and the lateral strain by ε_0, then, as is found from experiments, ε_0 is only a fraction of ε, i.e.,

$$\varepsilon_0 = \mu\varepsilon.$$

The factor μ is known as *Poisson's ratio*.

Poisson's ratio in tension is defined as

$$\mu = \frac{\varepsilon_0}{\varepsilon} = \frac{\text{lateral compressive strain}}{\text{longitudinal tensile strain}}$$

and in compression

$$\mu = \frac{\varepsilon_0}{\varepsilon} = \frac{\text{lateral tensile strain}}{\text{longitudinal compressive strain}}.$$

Poisson thought that the ratio μ was the same and equal to 0.25 for all materials. However, subsequent experiments showed that Poisson's ratio is different for different materials, ranging from 0 to 0.5. Average numerical values of this ratio for some materials are given in Table 2. In design practice μ is taken as 0.3 for steel; beyond the elastic limit μ increases to 0.5.

Table 2. Poisson's Ratio for Some Materials

Material	μ	Material	μ
Cork	0.00	Copper	0.34
Carbon steel	0.24 to 0.28	Bronze	0.35
Chrome-nickel steels	0.25 to 0.30	Rubber	0.47
Aluminium	0.26 to 0.36	Paraffine wax	0.5

Using this ratio, it is possible to determine the change in volume of a rod under tension or compression. Let us first solve this problem in the general form. The volume of a rod of square cross section before extension is

$$v_0 = a^2 l.$$

After extension each unit of the original length becomes equal to $(1+\varepsilon)$; consequently, the new length of the rod becomes equal to $l(1+\varepsilon)$. The unit of length in the transverse direction shortens and becomes equal to $(1-\varepsilon_0)$ or $(1-\mu\varepsilon)$. Therefore, the cross-sectional area after extension is $[a(1-\mu\varepsilon)]^2$.

The volume of the rod after extension is

$$v_1 = [a(1-\mu\varepsilon)]^2 l(1+\varepsilon)$$

or

$$v_1 = a^2 l(1+\varepsilon-2\mu\varepsilon-2\mu\varepsilon^2+\mu^2\varepsilon^2+\mu^2\varepsilon^3).$$

Neglecting terms containing the factors ε^2 and ε^3 as small quantities of higher order, we obtain

$$v_1 = a^2 l(1+\varepsilon-2\mu\varepsilon).$$

The increase in volume is

$$v_1 - v_0 = a^2 l(1+\varepsilon-2\mu\varepsilon) - a^2 l = a^2 l\varepsilon(1-2\mu).$$

The increase in unit volume is

$$\frac{v_1-v_0}{v_0} = \frac{a^2 l\varepsilon(1-2\mu)}{a^2 l} = \varepsilon(1-2\mu).$$

Since $\mu < 0.5$, then $1-2\mu > 0$ and the increase in volume is positive for all materials, i. e., the volume always increases under

tension. This is supported by experiment. It is only for paraffine wax (for which $\mu = 0.5$) that the volume remains unchanged.

Example 5. Determine the change in volume of a steel rod of length $l = 200$ mm and constant square section of side $a = 50$ mm subjected to a tensile force $P = 25$ tons if $\mu = 0.3$.

We first calculate the unit elongation ε. By Hooke's law (2.3) we have

$$\varepsilon = \frac{\sigma}{E} = \frac{P}{AE} = \frac{25{,}000}{5^2 \times 2 \times 10^6} = 0.0005.$$

Consequently,

$$v_1 - v_0 = a^2 l \varepsilon (1 - 2\mu) = 5^2 \times 20 \times 0.0005 (1 - 2 \times 0.3) = 0.1 \text{ cm}^3.$$

The increase in unit volume is

$$\frac{v_1 - v_0}{v_0} = \varepsilon (1 - 2\mu) = 0.0005 \times 0.4 = 0.0002 = 0.02\%.$$

8. Experimental Study of Materials in Tension

The design of structures calls for a knowledge of the properties of materials of which these structures are made.

The mechanical properties of materials are revealed by testing them under load.

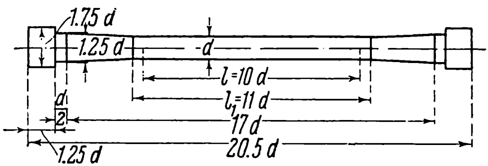


Fig. 14

The test most commonly used is a tension test. The reason for this is that the mechanical characteristics obtained from a tension test make it possible in many cases to predict sufficiently accurately the behaviour of the material under other types of deformation, such as compression, shear, torsion and bending. Besides, a tension test is easiest to perform.

Materials which have to withstand primarily compressive loads (stone, concrete, etc.) are tested in compression as well.

Tension tests are carried out on special specimens of materials in specially designed tension testing machines. Specimens are usually of circular section (Fig. 14), less frequently of rectangular section. At the ends of a specimen there are heads of heavier section.

The heads are inserted into special grips of the testing machine. The transition from the specimen head to the middle (gauge) length is made smooth, in the form of a cone in circular specimens and a fillet in flat specimens. Uniform extension of a specimen occurs over a distance where the specimen section is constant, therefore elongations are measured only over this distance, called the *gauge*

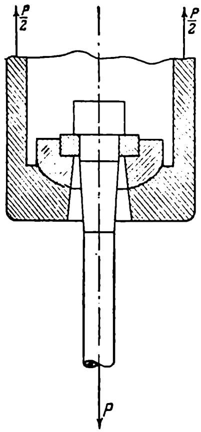


Fig. 15

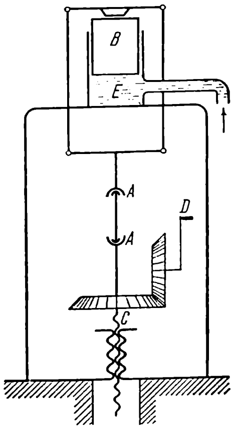


Fig. 16

length. In Fig. 14 the gauge length of the specimen is designated as l.

As experiments show, *only geometrically similar specimens of the same material give identical results*. Consequently, in comparing mechanical qualities of different materials the absolute dimensions of specimens may be different provided that the law of geometric similarity is maintained. In the case of brittle materials comparison is made by testing specimens of the same dimensions.

The shapes and dimensions of specimens are standardized; if, however, for some reason or other "normal" specimens cannot be prepared, comparable results may be obtained on specimens of circular or rectangular cross section similar to normal ones with the ratio

$$l:\sqrt{A}=11.3,$$

where l is the gauge length of the specimen and A its cross-sectional area.

This value of the ratio $l:\sqrt{A}$ for a circular specimen is obtained when $l = 10d$.

In order that a tensile force act precisely along the specimen axis, the gripping devices of the machine should be built with self-centring spherical seats (Fig. 15).

Tension testing machines subject a specimen to a load increasing gradually from zero to a value causing fracture, and provide the so-called static loading. The load is measured by load-measuring instruments (dynamometers).

Tension testing machines vary in construction. Figure 16 shows a schematic diagram of a machine widely used in materials testing laboratories. The heads A of a test specimen are held in the grips of the machine. The lower grip remains stationary during testing. It is raised or lowered only when the specimen is being mounted. The raising or lowering of the lower grip is effected by means of a screw C turning the handle D. The tensile force is produced by gradually pumping oil into a cylinder E mounted on the machine frame. Piston B moves up and lifts the upper grip through a system of pin-connected rods. Since the lower grip remains stationary during testing and the upper grip moves up, the specimen is stretched.

Tension testing machines are usually provided with recording instruments which trace a curve showing the relation between the tensile load and the resulting elongation of the specimen. As mentioned in Sec. 3, direct measurement of deformations is made with special instruments—strain gauges.

9. Tension Test Diagram and Its Characteristic Points

The behaviour of materials in tension is best understood from a consideration of a curve called a tension test diagram, which represents the stress-strain relation in tension. It is usually obtained from a diagram in the co-ordinates: tensile force P and absolute elongation of a specimen Δl. A P-Δl diagram is traced by a recording instrument or plotted from successive readings of the load and the corresponding increase in the length of the specimen. The forces measured at different instants during the testing are laid off to scale on the axis of ordinates, and the elongations on the axis of abscissas.

A diagram in these co-ordinates will, of course, depend on the dimensions of a specimen. The longer the specimen, the greater are the absolute elongations for one and the same force. In order to make these diagrams independent of the dimensions of test pieces and comparable for different materials, the ordinates should represent not forces but stresses σ obtained by dividing the tensile force

by the original cross-sectional area A_0 of the specimen

$$\sigma = \frac{P}{A_0}.$$

The abscissas should represent strains ε rather than absolute elongations.

The points of the tension test diagram thus obtained characterize the state of the specimen at different instants, and the entire diagram gives the stress-strain relation for the specimen over the duration of the test.

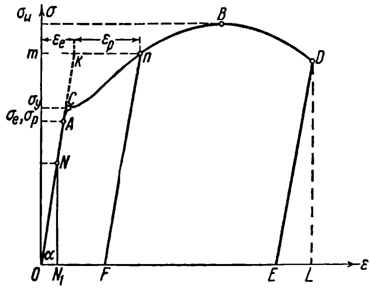


Fig. 17

Figure 17 shows a tension test diagram of mild steel. We proceed to consider its characteristic points.

Proportional limit. Up to a certain state characterized by point A in the diagram, the σ-ε relation is represented by a straight line. This is an indication of the fact that here elongations of the specimen increase directly as stresses. This straight line makes a very small angle with the axis of ordinates, i. e., the elongations of the specimen increase slowly in this portion. Point A corresponds to the stress known as the *proportional limit*. Up to the proportional limit Hooke's law holds good.

Consequently, *the proportional limit is defined as the maximum stress to which strains increase directly as stresses* in the material. The stress corresponding to the proportional limit is designated as σ_p.

If we consider any state of the specimen within the straight-line portion of the diagram, such as the state represented by point N (NN_1 is stress, ON_1 is strain), the slope of the straight-line portion to the axis of abscissas is given by the ratio

$$\tan\alpha = \frac{\overline{NN_1}}{\overline{ON_1}} = \frac{\sigma}{\varepsilon},$$

where σ is a concrete quantity and ε a pure number. On the other hand, according to Hooke's law (2.3)

$$\frac{\sigma}{\varepsilon}=E.$$

Consequently,

$$\tan\alpha=E,$$

i.e., the numerical value of the modulus of elasticity of the first kind can be determined, with the proper use of scales for the diagram, as the slope of the straight-line portion OA to the axis of abscissas.

Elastic limit. In designing a structure it is sometimes important to know the stress at which the material first undergoes plastic action. Extremely precise measurements show that even highly elastic materials develop permanent deformations under very small stresses. But the magnitude of these permanent deformations is so small that they are of no practical significance. Permanent deformations increase with increasing stress. *The elastic limit is defined as the stress at which the material develops a certain predetermined value of permanent strain* (0.002 to 0.005, or 0.2 to 0.5 per cent, of the original length of the specimen).

The elastic limit is designated as σ_e. The determination of the elastic limit presents great difficulties. It requires very precise and prolonged tests. In practice the magnitude of the elastic limit (for steel, for example) is very close to the proportional limit, and therefore point A corresponding to the proportional limit is considered to be coincident with the point corresponding to the elastic limit (as shown in Fig. 17). Further, as the stress increases, the tension test curve rises and departs from the straight line, turning smoothly to the right to point C.

Yield point (critical point). Some materials, such as mild steel, have a portion in the tension test diagram slightly above the proportional limit, from point C on, in which elongations begin to increase without increase in stress. This phenomenon is called yielding. *The yield point is defined as the stress at which a perceptible elongation occurs in the material without any increase of the stress.* The yield point is designated as σ_y. The point C of the diagram corresponding to the yield strength is called the *critical point*. Sometimes instead of a horizontal portion of the diagram there is even an inclined portion (sloping down to the right).

After passing the yield point the material recovers its ability to resist deformation but its elongation now begins to increase more rapidly than stresses; permanent deformations also increase rapidly. The yield point is a very important characteristic of the mechanical behaviour of a material since stresses above the yield point produce impermissible permanent deformations.

Many materials, such as alloy steels, have no pronounced yield point. The tension test diagram of such materials passes smoothly from the elastic part to a part where large permanent deformations occur. The yield strength of such materials is established in a pure conventional manner. The yield strength for them is considered as the stress at which they develop a permanent set (offset) equal to a specified value. Therefore, when speaking of the yield strength of such materials it is necessary to indicate the corresponding permanent set. The yield strength is commonly taken as the stress corresponding to a permanent set of 0.2 per cent.

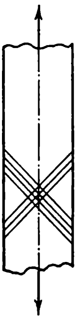


Fig. 18

When materials having a pronounced yield point are stretched, it is easy to observe the onset of yielding. If, for example, a tension testing machine is provided with a pointer indicating tensile forces, the pointer stops moving and remains on the same division for some time when the yield point of the material is reached though the deformations of the specimen continue to grow.

Also, the onset of yielding in the material can be noticed by observing the specimen itself. The polished surface of the specimen dulls and gradually becomes lustreless when the yield point is reached. Under close examination the surface exhibits lines inclined at about 45° to the axis of the specimen (Fig. 18). The number of these lines, known as Lüders lines, increases gradually and in consequence the surface of the specimen becomes dull. The occurrence of these lines and their propagation throughout the length of the specimen are evidence of the shears produced in crystals of the material.

Ultimate strength. Beyond the yield point the tension test diagram becomes curved (principally convex upward) and, as already stated, the deformations of the specimen begin to grow more rapidly than the stresses.

Point B (see Fig. 17) corresponds to the maximum value of the tensile force. The stress equal to the ratio of the maximum tensile force to the original cross-sectional area of the specimen is called the ultimate strength. The ultimate strength is designated as σ_u. After the ultimate strength is reached, a local reduction of area of the specimen, called *necking*, begins to occur gradually (Fig. 19). During necking the specimen elongates mainly at the necked-down portion while the remainder of the specimen elongates only slightly.

Since during necking the cross section at the neck becomes smaller and smaller, the deformation of the specimen occurs with decreasing load. The ultimate strength is a very important strength characteristic of a material, particularly important for brittle materials,

such as cast iron, hardened and cold-drawn steel, etc., which undergo relatively small deformations at fracture. At a stress corresponding to point D (see Fig. 17) the specimen ruptures. The stress at rupture lies below the ultimate strength in the tension test diagram. This is due to the fact that we agreed to calculate the stresses on the basis of the original cross-sectional area of the specimen. Actually, however, at the time of rupture the material develops the maximum stress since the area of the section aa (Fig. 19) becomes a minimum at that time. This stress is sometimes called the *true ultimate strength*.

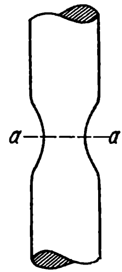


Fig. 19

The diagram considered above is termed an ordinary stress-strain diagram since the stresses are related to the original cross-sectional area and the elongations, to the original length. The cross section and length of the specimen vary continuously during the test. However, the ordinary diagram closely coincides with the true one up to the yield point. In the true diagram the ordinate is the stress obtained by dividing the force by the corresponding value of the minimum cross-sectional area of the specimen and the abscissa is the true unit elongation of the specimen, i. e., the change in length divided by the length of the specimen at the current instant.

Ductility of material. Besides the yield point and the ultimate strength characterizing the mechanical properties of a material, a very important characteristic is ductility of the material. The ductility of the material is characterized by the magnitude of the percentage elongation and the percentage reduction of the cross-sectional area at rupture.

The percentage elongation at rupture is expressed as

$$\delta = \frac{l_k - l_0}{l_0}\, 100\%,$$

where l_k is the length of the specimen after rupture and l_0 is the original length.

The percentage reduction of the cross-sectional area is found from the expression

$$\psi = \frac{A_0 - A_k}{A_0}\, 100\%,$$

where A_k is the cross-sectional area at the neck after rupture and A_0 is the original cross-sectional area of the specimen.

It is customary to distinguish between ductile and brittle materials depending on whether permanent deformations occurring in the specimen at rupture are large or small.

Figure 20 shows, for comparison, tension test diagrams of a ductile material (mild steel) and a brittle material (cast iron). It is seen that the brittle material fractures at a small strain and has no yield point. It should be noted, however, that the ductility of a material varies with the state of stress, strain rate, temperature and other conditions. A material exhibiting brittleness under tension at normal temperature may behave as a ductile material under other conditions, and conversely.

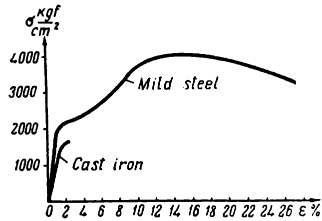


Fig. 20

Consider the deformation of the specimen beyond the elastic limit. If the specimen is unloaded at some point n of the diagram (Fig. 17) lying above the elastic limit, the line of unloading nF will be a straight line parallel to line OA. The segment mn represents the overall unit elongation of the specimen at the stress corresponding to point n. The segment OF equal to kn represents the amount of plastic deformation which remains in the specimen after unloading. The strain beyond the elastic limit is made up of two parts: the elastic strain which disappears after removal of the load and the plastic strain which remains after unloading the specimen

$$\varepsilon = \varepsilon_e + \varepsilon_p.$$

The elastic part of the strain beyond the elastic limit is proportional to the stress defined by segment Om.

Based on a so-called *law of unloading*, the elastic part of the strain can be determined beyond the elastic limit. Just before the rupture of the specimen its overall elongation is represented in the diagram by segment OL. After rupture the elastic part of the strain EL is recovered and only the permanent strain OE remains. The larger the permanent deformation, the more ductile is the material.

The mechanical properties of metals as revealed in tests depend on the chemical composition of the material, temperature, heat treatment, speed of testing, prestraining, etc.

The effect of chemical composition and heat treatment on mechanical properties is studied in metallography; here we shall briefly outline the effect of other factors on the mechanical properties of materials.

Temperature effect. The results of mechanical testing of materials usually relate to room temperature (15-20°C) at which tests are conducted in laboratories. However, many parts even of one and the same machine operate in widely different temperature conditions. Thus, the exhaust valves of an automobile engine operate

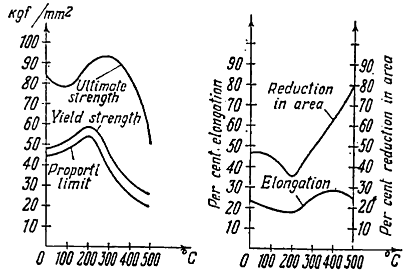


Fig. 21

at 500 to 800°C while engine parts which are in direct contact with the environment sometimes operate at very low temperatures.

For most materials the strength decreases and the ductility increases with increasing temperature. Mild steel behaves somewhat differently: at a temperature of about 250-300°C the ultimate strength of the steel attains a maximum value but falls off sharply with further increase in temperature.

Figure 21 presents diagrams showing the variation of the ultimate strength and ductility of steel with temperature. At high temperature, from 300-400°C, metals continue to deform, though very slowly, at constant load. Strain rate increases with increasing load or temperature. This property of metals to deform continually at constant load and high temperature is called *creep*.

Gas turbine blades operating at high temperature and subjected to centrifugal loads continually elongate with time. This elongation may cause the fracture of blades or dangerous brushing of these against the body, which sometimes happens in practice. Therefore, special steels and heat-resistant alloys exhibiting a small amount of creep are employed under these conditions.

At elevated temperatures the ultimate strength of a material depends also on the duration of testing. In these cases the strength of a material is referred to as *creep-rupture strength*. Figure 22 shows the creep-rupture strengths of a heat-resistant alloy at 700°C; as is seen, the strength of the material falls with increasing time of testing.

As the temperature drops off, the strength of steel increases but the ductility sharply decreases. At low temperature steel is very sensitive to all kinds of vibrations and blows (cold brittleness of steel). An addition of nickel increases its resistance to impact loads at low temperatures.

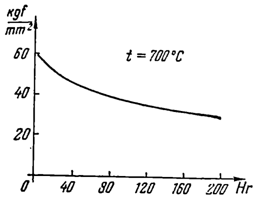


Fig. 22

Speed of testing. The mechanical characteristics of a material are also affected by the testing procedure. Therefore, to make the test results comparable it is necessary to follow a definite established testing procedure. Thus, for example, all metals possess the property of increasing their resistance to plastic deformation with increasing strain rate. Therefore, the more rapidly the specimen is loaded during testing, the higher are the resulting mechanical characteristics (proportional limit, yield point and ultimate strength) and the smaller the deformations. Steel possesses this property to a considerably lesser degree than more ductile metals such as zinc, lead, copper, etc.

The strain rate has its greatest effect on the yield point of a material. Under very rapid loading the yield stress may turn out to be higher than the ultimate strength obtained under slow loading. In view of this property of metals the rate of increase of stresses up to the yield point is usually not higher than 100 kgf/cm² per second under normal testing conditions.

10. Strain Hardening

If, prior to tension testing, a specimen of mild steel is loaded to a stress below the elastic limit and unloaded, the test diagram of the specimen will be no different from the tension test diagram of a specimen not subjected to preloading. If, however, a specimen is previously loaded to a stress above the yield point, the mechanical properties of the specimens being compared will be different.

Let a specimen of mild steel be stretched to a stress characterized by point A on the tension test diagram (Fig. 23). If the tensile load is now removed, a line AB is obtained in the

diagram, which is very close to a straight line. The elastic part of the overall elongation of the specimen disappears and a permanent elongation OB is observed. If the specimen is immediately stretched again, precise measurements will show that its proportional limit is lowered and the yield point is raised. The re-loading is represented by line BC in the diagram. The yield point becomes approximately equal to the stress to which the specimen was first stretched. If the specimen is allowed to "rest" for some time after unloading and then stretched, the proportional limit rises again, i. e., the material recovers its elasticity, and the yield point is raised to a still greater extent (dashed line $CC'D'$).

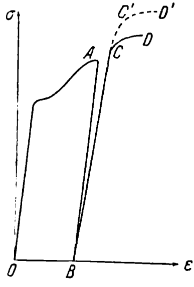


Fig. 23

The complete recovery of elastic properties requires a certain length of time which depends on the kind of material.

An increase in strength and a loss in ductility because of pre-stretching beyond the yield point are termed *work hardening* or *strain hardening*. Strain hardening changes the mechanical qualities of a material and residual stresses are set up in the material. In some cases the phenomenon of strain hardening is undesirable and should be controlled, while in other cases strain hardening is artificially produced.

When holes are punched in a sheet for rivets, the material at the edge of a hole undergoes strain hardening and becomes stiffer. This promotes the formation of cracks.

To avoid the detrimental effect of strain hardening in this case, the strain-hardened portion of the material is removed, increasing the hole diameter by drilling. The effect of strain hardening can also be eliminated by annealing, i. e., by heating the material to a certain temperature, holding it at that temperature for some time and subsequently slowly cooling. In other cases, as already stated, strain hardening is artificially produced. For instance, chains of lifting machines are pre-stretched above the yield point to make them less ductile and to avoid large deformations during operation, which would prevent the entry of the chain links in their seats on the drum.

Strain hardening is responsible for the fact that a wire obtained by drawing has a considerably higher strength than the steel from which it is made.

11. Strain Energy in Tension

Take a tension test diagram in the co-ordinates P and Δl (Fig. 24) and see what is represented by the total area under the diagram $OABDE$. Since the abscissa is the total elongation of the specimen or, in other words, the distance travelled by the point of application of the tensile force, and the ordinate is the magnitude of this force, *the total area under the diagram* $OABDE$ obviously *represents*

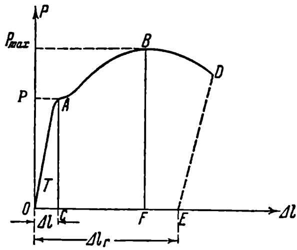


Fig. 24

the work done by the external tensile force in breaking the specimen. If the load on the specimen stretched above the elastic limit is gradually released, the specimen begins to contract gradually but it can no longer recover its original dimensions completely. Consequently, the work done in stretching the specimen beyond the elastic limit will not be given up completely; part of the work done by the external forces is expended in producing permanent elongations.

The area OAC under the initial portion of the diagram up to the elastic limit represents the elastic strain energy which is stored in the material as potential energy and can be given up completely after the load is removed. Denoting this energy by U, the load corresponding to the elastic limit by P and elongation produced by Δl, we find

$$U = \frac{P\Delta l}{2}; \tag{2.5}$$

since

$$\Delta l = \frac{Pl}{EA},$$

we have

$$U = \frac{P^2 l}{2EA} \tag{2.6}$$

or, replacing P by its expression from formula (2.4), we write

$$U = \frac{\Delta l^2 EA}{2l}. \tag{2.7}$$

The strain energy within the elastic limit can also be expressed in terms of stress. Since

$$P = \sigma A,$$

we obtain from formula (2.6)

$$U = \frac{\sigma^2 Al}{2E}. \tag{2.8}$$

If the strain energy is estimated per unit volume, we obtain the *elastic strain energy per unit volume*

$$u = \frac{1}{2}\frac{\sigma^2}{E} = \frac{\sigma\varepsilon}{2}. \tag{2.9}$$

Formulas (2.5) through (2.9) can be used to calculate the strain energy for any point of the diagram within the elastic limit.

Example 6. Calculate the elastic strain energy per unit volume due to extension of a steel having the elastic limit $\sigma_e = 2{,}500$ kgf/cm² and the modulus of elasticity $E = 2\times10^6$ kgf/cm².

Solution. The elastic strain energy per unit volume is determined by formula (2.9)

$$u = \frac{\sigma^2}{2E} = \frac{2{,}500^2}{2\times2\times10^6} = 1.56 \text{ cm-kgf/cm}^3.$$

Example 7. Compare the strain energies of two steel bars of circular section (Fig. 25) if the maximum tensile stresses in both bars are the same and equal to

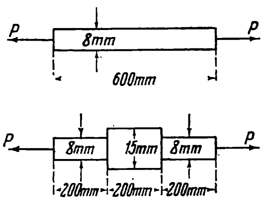


Fig. 25

$$\sigma = 1{,}500 \text{ kgf/cm}^2, \qquad E = 2.2\times10^6 \text{ kgf/cm}^2.$$

Solution. Since the minimum sections and the maximum stresses in both bars are the same, the tensile forces must be equal as well. From formula (2.2) we have

$$P = \sigma A = 1{,}500\times\frac{\pi}{4}\times0.8^2 = 754 \text{ kgf}.$$

The strain energy of the first bar is, according to formula (2.6),

$$U_1 = \frac{P^2 l}{2EA} = \frac{754^2\times60}{2\times2.2\times10^6\,\frac{\pi}{4}\,0.8^2} = 15.4 \text{ cm-kgf}.$$

The strain energy of the left-hand and right-hand parts of the second bar is

$$U_2' = \frac{2}{3} U_1 = \frac{2}{3} \times 15.4 = 10.27 \text{ cm-kgf}.$$

The strain energy of the middle part of the second bar is

$$U_2'' = \frac{P^2 l}{2EA} = \frac{754^2 \times 20}{2 \times 2.2 \times 10^6 \frac{\pi}{4} 1.5^2} = 1.46 \text{ cm-kgf}.$$

The total strain energy of the second bar is

$$U_2 = U_2' + U_2'' = 10.27 + 1.46 = 11.73 \text{ cm-kgf}.$$

Consequently,

$$\frac{U_1}{U_2} = \frac{15.4}{11.73} \cong 1.4.$$

Thus, the bar of uniform section, having the same length and the same maximum stress as the stepped bar, absorbs 1.4 times as much energy as the stepped bar. This must be regarded as a favourable property, particularly under impact loading.

12. Compression Testing

Comparative tension and compression tests on steels show that stress-strain relations are identical up to stresses corresponding to large plastic deformations. Therefore, steels are rarely tested in compression. In special cases such as the fabrication of roller and ball bearings, steels must be tested in compression. Cast iron, which acts primarily in compression and bending, is more often tested under these types of deformation and less frequently under tension.

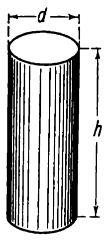


Fig. 26

To test metals in compression, specimens are made in the form of cylinders (Fig. 26) for which $1 < h/d < 3$. Specimens for testing stone, cement, and wood are often of cubic shape. Compression tests are conducted in special presses or universal machines which allow materials to be tested in both tension and compression. The shape assumed by the specimen during testing depends on the kind of material, the ratio of the height to cross-sectional dimensions of the specimen and mainly on the friction developed over the contact areas between the specimen end surfaces and the apparatus compression plates.

Considerable friction at the ends of the specimen results in a non-uniform stress distribution over cross sections. Therefore, care

must be taken to reduce the frictional effects by lubricating the specimen end surfaces or inserting special gaskets of softer material than the test material. The barrel shape of compression specimens

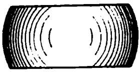


Fig. 27

Fig. 28

(Fig. 27) and the typical mode of fracture of a cement cube (Fig. 28) are due exclusively to the friction at the ends of the specimen.

Ductile materials do not fracture when tested in compression. A specimen of this kind of material flattens and assumes the shape of a disk. As the cross-sectional area increases the resistance to deformation increases, therefore the shape of the compression test diagram beyond the yield point is different from that of the tension test diagram.

Figure 29 shows compression test diagrams for a ductile material (mild steel) and a brittle material (cast iron). Ductile materials, such as mild steel, have no ultimate compressive strength.

As is seen from the diagram, a brittle material, such as cast iron, fractures under compression with very small strain.

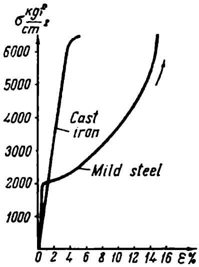


Fig. 29

Table 3 gives ultimate strengths and percentage elongations at fracture for some materials.

13. Hardness

Hardness is defined as the ability of a material to resist the penetration of another, harder object into it. Hardness is an important property of a material; besides, hardness enables an estimate of the strength of a material to be made.

There are several methods available for the determination of hardness of metals. The method most commonly used is pressing a quenched steel ball into the test metal.

If a quenched steel ball of diameter D is pressed into the test metal under a load P, as shown in Fig. 30, the ball penetrates into the metal to a depth h and leaves an impression in the form of a

Table 3. Ultimate Strength and Percentage Elongation

Material	σ_u, kgf/mm²	δ%	Material	σ_u, kgf/mm²	δ%
Steel, 10	32-40	28	Oak, parallel to grain	9.5 (tension) 5 (compression)	4
Steel, 20	40-50	26	Pine, parallel to grain	8 (tension) 4 (compression)	3
Steel, 30	48-60	22	Beech	13 (tension)	3
Steel, 40	60-75	20	Granite	0.5-0.8 (tension) 4-25 (compression)	— —
Steel, 50	63-80	16	Sandstone	0.25 (tension) 4-15 (compression)	—
3pct nickel steel	78	24	Brick	0.74-3 (compression)	—
Chrome-nickel steel (hardened)	115-140	12			
Spring steel (hardened)	135-155	6-8			
Aluminium, drawn	9-10	8-13			
Copper	22	35-38			
Gray cast iron, ordinary	14-18 (tension) 60-100 (compression)	—			

circular depression of diameter d because of plastic deformation of the test metal. The quantity characterizing hardness or what is known as the Brinell hardness number (H_B) is the ratio of the load P, under which the ball is pressed, to the surface area A of the impression left on the test metal after indentation

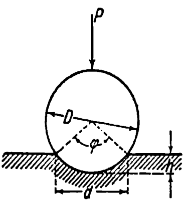


Fig. 30

$$H_B=\frac{P}{A}=\frac{P}{\frac{\pi D}{2}\left(D-\sqrt{D^2-d^2}\right)}. \qquad (2.10)$$

Hardness tests are often conducted on finished work-pieces. The thickness of work-pieces may be different. The smaller the thickness, the smaller the diameter of the ball required and the smaller the force applied to press it against the part. To obtain equal hardness numbers for one and the same material, the pressures on the balls must be proportional to the squares of the ball diameters, i. e.,

$$\frac{P_1}{P_2}=\frac{l_1^2}{D_2^2}=\frac{d_1^2}{d_2^2}.$$

The diameters of balls vary with the thickness of the work-piece to be tested.

For steels the relation between the hardness number and the ultimate tensile strength is approximately expressed as

$$\sigma_{u.t} \cong 0.36 H_B. \quad (2.11)$$

Table 4 gives Brinell hardness numbers for some materials.

Table 4

Material	Brinell hardness number, H_B (kgf/mm²)	Material	Brinell hardness number, H_B (kgf/mm²)
Steel	150-300	Copper	60
Steel, hardened	up to 850	White metal (for bearings)	20-28
Cast iron	130-300		
Aluminium, rolled	45		

14. Check Questions

State the hypothesis of plane sections.

What is the total or absolute elongation?

What is percentage elongation and its dimension?

State Hooke's law; how is it expressed mathematically?

What does the modulus of elasticity of the first kind characterize?

What is the dimension of the modulus of elasticity?

Do all materials obey Hooke's law?

What is the stiffness of a bar in tension and compression?

What is Poisson's ratio?

What are the characteristic points of the tension test diagram for mild steel?

What are the proportional limit, the elastic limit, the yield point, the ultimate strength?

How can the modulus of elasticity E be determined from the tension test diagram?

When do Lüders lines appear on a test specimen?

Why does the stress at which fracture occurs lie below the ultimate strength on the tension test diagram?

What is the ductility of a material? What characterizes it?

State the law of unloading.

What is the strain hardening of a material?

What is represented by the area under the tension test diagram?

What is strain energy per unit volume and its dimension?

How do the mechanical properties of steel vary with increasing and decreasing temperature?

How are the mechanical characteristics affected by the rate of loading?

What is hardness and how is it measured?

What is the hardness number?

When does a specimen begin to neck down?

Compare the elastic strain energy per unit volume for 10 steel and spring steel assuming $\sigma_e = 0.5\sigma_u$.

Chapter III

Strength design for tension and compression

15. Allowable Stress and Selection of Sections

The first problem the designer is confronted with is the choice of material. As stated above, the choice of material is governed primarily by the operating conditions of structural elements to be designed. Economy considerations and technology of manufacture are also taken into account in choosing the material. However, this is inadequate for a rational choice of material. In the earlier discussion we stressed the marked difference in the behaviour of ductile and brittle materials in tension and compression tests. We shall now discuss one more point which must be taken into account in choosing the material. The point is that, in distinction to brittle materials, ductile materials behave quite differently with respect to so-called localized stresses, i. e., stresses arising on a very small portion of the cross section and considerably exceeding the stresses on the remainder of the section.

In the case of tension or compression the stresses are uniformly distributed over the cross section only in prismatic bars of constant section. It is difficult, however, to indicate any one part of a machine which would represent a bar of constant section. Even in such a simple part as a bolt the cross section may change abruptly, as in the threaded portion of the bolt or in the transition from bolt pin to head. Machine parts usually fail where the cross section changes abruptly. This reduction in strength is due to high localized stresses in the region of abrupt changes of cross-sectional dimensions. Thus, in a circular tension specimen with a groove (Fig. 31) or a specimen of rectangular section with a hole (Fig. 32) the stresses are not uniformly distributed over the dangerous cross section but in the way shown in the relevant figures.

An abrupt increase in the stresses occurs in a small zone and the stresses diminish rapidly with the distance from this zone.

The ratio of the maximum localized stress to the average stress which would result if the stresses were uniformly distributed is called the *stress concentration factor*. Denoting the stress concentration factor by α, we have

$$\alpha = \frac{\sigma_{max}}{\sigma}, \tag{3.1}$$

where σ_{max} is the maximum localized stress and σ is the stress in the case of uniform stress distribution, i. e., the average stress; this stress is often referred to as *nominal stress*. For a tension bar the nominal stress is

$$\sigma = \frac{P}{A},$$

where A is the cross-sectional area at the location in question.

To find the magnitude of the stress concentration factor, it is obviously necessary to know how to determine the magnitude of localized stresses, σ_{max}. This problem is very difficult and is not

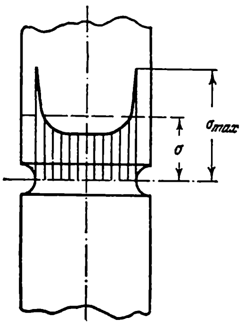


Fig. 31

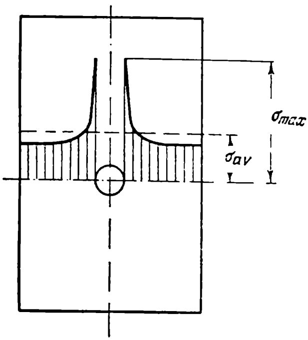


Fig. 32

solvable by the methods of strength of materials. In most cases the maximum stresses are determined by the methods of the theory of elasticity or experimentally. For many typical cases of stress concentration (grooves, drilled holes, fillets, keyways, thread, etc.) stress concentration factors have been determined under different types of deformation. The values of these stress concentration factors may be found in handbooks. They usually lie between 1.2 and 3.

Stress concentration is of great importance in choosing the material under various types of loading. Dividing the materials into "ductile" and "brittle" according to whether the material fails with appreciable or small elongation under static tension and normal temperature, it may be said that stress concentration differently affects the strength of ductile and brittle materials. Take a specimen with stress concentration, which is made of a sufficiently ductile material, and subject it to static extension. As the load

increases, so do the stresses in the specimen. After the maximum stress in the region of stress concentration reaches the yield stress, it will no longer grow with increasing load. The stresses will grow with load where they have not yet reached the yield stress. Thus, with further increase in load the stresses will level off along the length and across the section of the specimen. Therefore, it may be said that stress concentration does not reduce the strength of a ductile material under static loading and, consequently, in this case stress concentration may be disregarded in design. It is different in the case of a brittle homogeneous material, such as high-strength brittle steel. Here stress concentration cannot be neglected.

Modern engineering employs so many materials with different mechanical properties to suit different operating conditions that even a brief survey of mechanical properties of these materials would require too much space. Comprehensive data on these properties as provided by mechanical tests may be found in the appropriate handbooks.

Mechanical testing of materials gives limiting stress values (ultimate strength, yield strength) which, when reached in machine parts, entail either fracture or such large deformations that they are impermissible.

For safe service of a structure, the stresses induced in its elements must be lower than these limiting values. Therefore, the second important point in design is the choice of the safe or so-called *allowable stress*. The allowable stress is defined as the maximum stress at which the strength and a certain duration of service of the structural element to be designed are ensured. The allowable stresses are only a certain fraction of the limiting stresses. The number which expresses the ratio of the limiting stress to the allowable stress is called the *factor of safety*.

Depending on the type of loading and material, a certain limiting stress is taken as a basis for the choice of the allowable stress.

In the case of brittle materials, the starting limiting stress is taken as the ultimate strength. In this case the allowable stress is

$$[\sigma]=\frac{\sigma_u}{k}\,. \tag{3.2}$$

If a structural element of a very brittle steel involves a localized stress and this stress is the maximum one for the member, the allowable stress is then determined by the formula

$$[\sigma]=\frac{\sigma_u}{\alpha k}\,, \tag{3.3}$$

where α is the stress concentration factor, which is taken from the handbook.

When stress concentration is present in a part, the material chosen must be ductile, low-sensitive to localized stresses.

If a structural element is made of a non-homogeneous material, such as cast iron, the stress concentration factor is disregarded. The point is that in such materials the stress concentration caused by an abrupt change of the section is less than that created by the non-homogeneity of the material.

In the case of ductile materials, such as structural steel, the starting limiting stress is taken as the yield strength. In this case the allowable stress is

$$[\sigma]=\frac{\sigma_y}{k}. \tag{3.4}$$

For a constant load and a ductile material the stress concentration caused by an abrupt change of sections is neglected; therefore, in the presence of stress concentration the allowable stress is determined again by formula (3.4).

The assignment of the allowable stress or the factor of safety is of great practical importance. If the allowable stress is chosen too high, the structure will not be strong enough. On the other hand, if the allowable stress is too low, the dimensions of the structure will be too large, which entails an increase in weight and cost of the structure. In many cases overweight cannot be tolerated at all, as in aircraft construction.

The proper allowable stress can be chosen only by consideration of various influencing factors. First of all, it should be noted that external loads which will act on the structural element to be designed cannot be determined accurately in many cases. Besides, the stresses are often determined only approximately, especially in cases of localized stresses and complex-shaped structural elements. The lower the accuracy with which the loads and stresses are estimated, the larger factor of safety should be provided in assigning the allowable stress.

The factor of safety should be assigned so as to balance our inexact knowledge of loads and stresses in design. The magnitude of the allowable stress is also affected by the type of material used: the less homogeneous the material, the larger factor of safety should be taken in assigning the allowable stress since the mechanical characteristics of a non-homogeneous material cannot be determined with accuracy.

The factor of safety is the larger, the longer the probable lifetime of the structure. The factors of safety for parts of an aircraft engine are considerably smaller than for stationary engines since the weight of aircraft engines should be minimum, but the duration of service of aircraft engines is much shorter than that of stationary engines.

The foregoing general considerations which govern the choice of allowable stresses under all types of deformation show the complexity of this problem. It is impossible to give general specifications of allowable stresses suitable for all cases encountered in practice; it is particularly difficult to give such specifications for all fields of mechanical engineering. In some fields of mechanical and civil engineering such specifications exist; their use is obligatory in these fields. With the improvement of design methods, the accumulation of experience, the extension of knowledge of material properties the specifications of allowable stresses are supplemented and amended from time to time.

In choosing allowable stresses in cases where no specifications exist, one is guided by the foregoing general considerations and the experience with operation of previously designed similar structures.

In Table 5 are given tentative values of allowable stresses in tension and compression for some materials under static loading.

At present there is a swing from the design based on allowable stresses to an improved procedure including in part the foregoing considerations; some explanations will be given in Sec. 20.

Table 5. Tentative Values of Basic Allowable Stresses in kgf/cm^2

<table>
<tr><th>Material</th><th>$[\sigma_t]$</th><th>$[\sigma_c]$</th></tr>
<tr><td>Grey pig iron</td><td>—</td><td>1,200-1,500</td></tr>
<tr><td>Steel, OC and 20</td><td colspan="2">1,400</td></tr>
<tr><td>Steel, 30</td><td colspan="2">1,800</td></tr>
<tr><td>Steel, 30, in bridges</td><td colspan="2">1,600</td></tr>
<tr><td>Structural carbon steel in engineering</td><td colspan="2">600-2,500</td></tr>
<tr><td>Structural alloy steel in engineering</td><td colspan="2">1,000-4,000 and higher</td></tr>
<tr><td>Copper</td><td colspan="2">300-1,200</td></tr>
<tr><td>Brass</td><td colspan="2">700-1,400</td></tr>
<tr><td>Bronze</td><td colspan="2">600-1,200</td></tr>
<tr><td>Aluminium</td><td colspan="2">300-800</td></tr>
<tr><td>Albronze</td><td colspan="2">800-1,200</td></tr>
<tr><td>Duralumin</td><td colspan="2">900-1,600</td></tr>
<tr><td>Elektron</td><td colspan="2">900-1,200</td></tr>
<tr><td>Cloth laminate</td><td colspan="2">300-400</td></tr>
<tr><td>Hardened paper</td><td colspan="2">500-700</td></tr>
<tr><td>Bakelized plywood</td><td colspan="2">400-500</td></tr>
<tr><td>Pine, parallel to grain</td><td>80-100</td><td>100-120</td></tr>
<tr><td>Pine, perpendicular to grain</td><td>—</td><td>15-20</td></tr>
<tr><td>Oak, parallel to grain</td><td>100-130</td><td>130-150</td></tr>
<tr><td>Oak, perpendicular to grain</td><td>—</td><td>20-35</td></tr>
<tr><td>Masonry</td><td>up to 3</td><td>8-40</td></tr>
<tr><td>Brickwork</td><td>up to 2</td><td>6-20</td></tr>
<tr><td>Concrete</td><td>3-10</td><td>40-200</td></tr>
</table>

Tensile or compressive stresses are determined by formula (2.2):

$$\sigma = P/A.$$

If the allowable stress is introduced, the static strength equation or the design equation for tension and compression becomes

$$\sigma = \frac{P}{A} \leqslant [\sigma]. \tag{3.5}$$

where $[\sigma]$ is the allowable stress in tension or compression.

This design equation enables one to solve the following problems:

(1) Given a force P and an allowable stress $[\sigma]$, determine the required cross-sectional area

$$A \geqslant \frac{P}{[\sigma]}. \tag{3.6}$$

(2) Given a cross-sectional area and an allowable stress, determine the allowable load

$$P \leqslant [\sigma] A. \tag{3.7}$$

(3) Given a force P and a cross-sectional area A, determine whether the body is strong enough by comparing the stress σ found from formula (1.3) with the allowable stress $[\sigma]$.

Example 8. A hollow cast iron cylindrical support of length 25 cm is compressed by a load $P = 8$ tons. Determine the outer diameter D, the inner diameter d and the magnitude of the total contraction of the support if the allowable stress in compression is $[\sigma] = 1{,}200\ \text{kgf/cm}^2$, $E = 8 \times 10^5\ \text{kgf/cm}^2$ and $d:D = 4:5$.

Solution. From Eq. (3.5) we have

$$\frac{P}{\frac{\pi}{4}\left[D^2 - \left(\frac{4}{5}D\right)^2\right]} = \frac{100P}{9\pi D^2} \leqslant [\sigma],$$

whence

$$D \geqslant \sqrt{\frac{100P}{9\pi[\sigma]}} = \sqrt{\frac{100 \times 8{,}000}{9 \times 3.14 \times 1{,}200}} = 4.8\ \text{cm}.$$

Take $D = 5$ cm. The inner diameter is

$$d = \frac{4}{5}D = \frac{4}{5} \times 5 = 4\ \text{cm}.$$

The absolute contraction of the support is found from formula (2.4)

$$\Delta l = \frac{Pl}{EA} = \frac{8{,}000 \times 25}{8 \times 10^5\ \frac{\pi}{4}(5^2 - 4^2)} = 0.036\ \text{cm}.$$

Example 9. In the case of slightly stressed chains the diameter of the chain iron (Fig. 33) is determined using the formula

$P=1{,}000d^2$ (where P is the tensile force in kgf, d is the diameter of the shank of the chain link in cm). Determine the allowable stress adopted in this formula.

Solution. The design equation for a chain link is

$$\frac{P}{2\frac{\pi d^2}{4}} \leqslant [\sigma], \quad \text{whence} \quad P \leqslant [\sigma]\frac{\pi}{2}d^2.$$

From comparison of the above expression for the tensile force P with the given formula we find

$$[\sigma]\frac{\pi}{2}d^2 \geqslant 1{,}000d^2,$$

whence

$$[\sigma] \geqslant \frac{1{,}000}{\frac{\pi}{2}} = 637 \text{ kgf/cm}^2.$$

Example 10. A cast iron bracket ABC (Fig. 34) carries a load $Q=5$ tons suspended from the hinge. Determine the required cross-

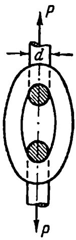


Fig. 33

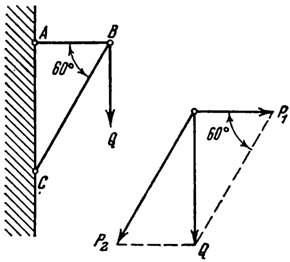


Fig. 34

sectional areas of rod AB and boom BC if the allowable stresses for cast iron in tension is 300 kgf/cm² and in compression 900 kgf/cm².

Solution. Resolve the force Q into two components along rod AB and boom BC.

The force stretching the rod is

$$P_1 = \frac{Q}{\tan 60^\circ} = \frac{5{,}000}{1.732} = 2{,}880 \text{ kgf}.$$

The force compressing the boom is

$$P_2 = \frac{Q}{\sin 60^\circ} = \frac{5{,}000}{0.866} = 5{,}780 \text{ kgf}.$$

The required cross-sectional area of the rod is

$$A_1 \geqslant \frac{2,880}{300} = 9.6 \text{ cm}^2 \cong 10 \text{ cm}^2.$$

The required cross-sectional area of the boom is

$$A_2 \geqslant \frac{5,780}{900} = 6.43 \text{ cm}^2 \cong 6.5 \text{ cm}^2.$$

16. Effect of Gravity in Tension and Compression

In cases where the weight of a body to be designed is small compared with the external load, the effect of gravity is neglected in strength design. But when the length of a rod is considerable (connecting rods, ropes, chains) or when engaged in the design of walls, stone bridge abutments, etc., the deadweight can no longer be neglected: it must be introduced into the design as an additional load contributing to the stress.

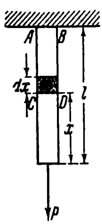


Fig. 35

Consider a long rod (Fig. 35) subjected to a tensile force P. The most dangerous section of the rod is the section AB at the fixed end. At this section the rod is stretched by the force P and the total weight of the rod W, i. e., by the force $P+W$. Denote the cross-sectional area of the rod by A; the maximum stress at the critical section of the rod is then

$$\sigma_{max} = \frac{P+W}{A}. \tag{3.8}$$

Substituting this expression in Eq. (3.5), we obtain the design equation

$$\sigma_{max} = \frac{P+W}{A} \leqslant [\sigma]. \tag{3.9}$$

Since the weight of the rod is $W = Al\gamma$, where γ is the weight per unit volume of the material, the design equation may be rewritten as follows

$$\frac{P}{A} + l\gamma \leqslant [\sigma]. \tag{3.10}$$

Hence the required cross-sectional area is

$$A = \frac{P}{[\sigma] - l\gamma}. \tag{3.11}$$

The stress due to gravity at an arbitrary section CD a distance x from the lower end is

$$\sigma_x = \frac{W_x}{A} = \frac{Ax\gamma}{A} = x\gamma. \tag{3.12}$$

From this expression it is seen that the stress in a rod of uniform section due to its own weight is independent of the cross-sectional area. Besides, the same expression shows that, for a length x such that the quantity $x\gamma$ becomes equal to the ultimate tensile strength of the material $\sigma_{u.t}$, the rod breaks due to its own weight. The length for which a rod breaks due to its own weight is termed the *critical length*, l_{cr}. On the basis of expression (3.12) we have

$$l_{cr} = \frac{\sigma_{u.t}}{\gamma}. \tag{3.13}$$

The length for which the allowable stress is reached in a rod due to its own weight alone is called the *limiting length*, $l_{\lim}$

$$l_{\lim} = \frac{[\sigma]}{\gamma}. \tag{3.14}$$

In determining the maximum stress [formula (3.8)] we found the total force acting at the most dangerous section of the rod and then obtained the stress produced by this force. But we could have found the maximum stress in a different way, namely, by determining the stress due to the force P and the stress caused by the weight of the rod W separately and then adding these stresses. As can readily be seen, the result would be the same. In the latter case we would use the so-called *principle of superposition*. This principle states that when a system is acted on by several loads the stresses or strains can be determined as the sum of the stresses or strains caused by each load acting separately.

Thus, it turns out that each of the loads taken separately produces the same effect as if it were the only load acting on the system. The principle of superposition is valid when the total stress or the total strain due to all loads remains within Hooke's law. Otherwise the principle cannot be used. In some cases this principle facilitates the solution of problems of strength of materials.

We shall use the principle of superposition to determine the deformation of the rod (see Fig. 35). Determine first the deformation of the rod due to its own weight. Isolate an element of length dx from the rod at a distance x from the lower end by two infinitely close cross sections (Fig. 35). Since the length of the isolated element is infinitesimally small, the tensile force in the length dx may be considered constant. This force is equal to $A\gamma x$. The absolute elongation of the isolated element is, from Hooke's law (2.4),

$$\Delta(dx) = \frac{A\gamma x\,dx}{EA} = \frac{\gamma}{E}x\,dx,$$

Integrating this expression from 0 to l, we obtain the elongation of the whole rod

$$\Delta l = \int_0^l \frac{\gamma}{E} x\,dx = \frac{\gamma l^2}{2E}. \tag{3.15}$$

Expression (3.15) may be represented in an alternate form, substituting W/A for γl

$$\Delta l = \frac{Wl}{2EA}. \tag{3.16}$$

The elongation of the rod produced by the force P is Pl/EA.

From this it is seen that the elongation of the rod due to its own weight is half the elongation produced by a force equal to the weight of the rod and applied to its end. Using the principle of superposition we find the total elongation of the rod

$$\Delta l = \frac{Pl}{EA} + \frac{Wl}{2EA} = \frac{\left(P + \frac{W}{2}\right) l}{EA}. \tag{3.17}$$

Example 11. Determine the limiting and critical length of a steel bar if the allowable stress is $[\sigma] = 1{,}500$ kgf/cm², the ultimate tensile strength $\sigma_{u.t} = 4{,}500$ kgf/cm² and the specific weight $\gamma = 0.0078$ kgf/cm³.

Solution. The limiting length is

$$l_{\lim} = \frac{[\sigma]}{\gamma} = \frac{1{,}500}{0.0078} = 192{,}000 = 1{,}920 \text{ m}.$$

The critical length is

$$l_{cr} = \frac{\sigma_{u.t}}{\gamma} = \frac{4{,}500}{0.0078} = 576{,}000 = 5{,}760 \text{ m}.$$

Example 12. Determine the stress in and the elongation of a steel rod of square section 5×5 cm and length $l = 10$ m produced by its own weight and a tensile load $P = 15$ tons if $E = 2 \times 10^6$ kgf/cm² and the specific weight is $\gamma = 7.8$ gf/cm³ $= 0.0078$ kgf/cm³.

Solution. The maximum stress in the rod is

$$\sigma_{\max} = \frac{P}{A} + l\gamma = \frac{15{,}000}{25} + 1{,}000 \times 0.0078 = 600 + 7.8 \cong 608 \text{ kgf/cm}^2.$$

The stress induced in the rod, if its own weight were neglected, would be 600 kgf/cm²; consequently, the neglect of the weight of the rod would introduce an error in the stress equal to

$$\frac{608 - 600}{608} \times 100 \cong 1.2\%.$$

The elongation, with the weight of the rod taken into account, is determined by formula (3.17)

$$\Delta l=\frac{Pl}{EA}+\frac{Wl}{2EA}=\frac{15{,}000\times 1{,}000}{2\times 10^6\times 25}+\frac{25\times 1{,}000\times 0.0078\times 1{,}000}{2\times 2\times 10^6\times 25}=$$
$$=0.3+0.00195=0.30195 \text{ cm}.$$

The neglect of the weight of the rod would introduce an error in the elongation equal to

$$\frac{0.30195-0.3}{0.30195}\times 100=0.65\%.$$

As can be seen from this example, when the length of a rod is small the neglect of its own weight involves only a small error.

17. Stepped Rod

From formula (3.12) it is seen that a long prismatic rod has different stresses at different sections. In such rods the distribution of material is not economical. Assigning the rod, throughout its length, equal cross-sectional dimensions, as found by consideration of the most severely stressed section, would unnecessarily increase the weight of the rod. To reduce the effect of the weight and to utilize the material more rationally in long rods, ropes, tall bridge piers, towers, they are made of variable section. A rod can be shaped so that the stresses at all of its cross sections will be the same. Such a rod is called a *rod of uniform strength*. Because of complexity of fabrication, rods of uniform strength are sometimes replaced by stepped rods.

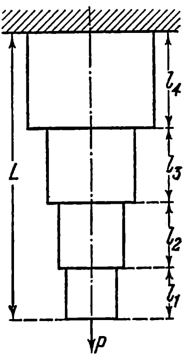


Fig. 36

Derive a formula for determining the cross-sectional areas of a stepped rod (Fig. 36) stretched by a force P and its own weight. Neglect the stress concentration at abrupt changes of cross sections.

In the general case the lengths of individual elements of the rod may be different: l_1, l_2, l_3, ..., l_n. The first, lowest, prismatic element of the rod is stretched by the force P and the weight of this element. The required cross-sectional area of this element is determined from formula (3.11)

$$A_1\geqslant\frac{P}{[\sigma]-l_1\gamma}.$$

To the bottom section of the second element is applied the force $P+A_1l_1\gamma$ or $A_1[\sigma]$. To determine the required cross-sectional area of the second element we use again formula (3.11), substituting $A_1[\sigma]$ for P

$$A_2 \geqslant \frac{A_1[\sigma]}{[\sigma]-l_2\gamma}.$$

Substituting the value of A_1, we obtain

$$A_2 \geqslant \frac{P[\sigma]}{([\sigma]-l_1\gamma)([\sigma]-l_2\gamma)}.$$

To the bottom section of the third element is applied the force $A_2[\sigma]$; consequently, by the same formula (3.11)

$$A_3 \geqslant \frac{A_2[\sigma]}{[\sigma]-l_3\gamma} = \frac{P[\sigma]^2}{([\sigma]-l_1\gamma)([\sigma]-l_2\gamma)([\sigma]-l_3\gamma)}.$$

Similarly, we write a formula for the cross-sectional area of the nth element

$$A_n \geqslant \frac{P[\sigma]^{n-1}}{([\sigma]-l_1\gamma)([\sigma]-l_2\gamma)([\sigma]-l_3\gamma)\dots([\sigma]-l_n\gamma)}.$$

In the particular case when the lengths of all portions are the same, i.e., $l_1=l_2=l_3=\dots=l_n=\frac{L}{n}$

$$A_n \geqslant \frac{P[\sigma]^{n-1}}{\left([\sigma]-\frac{L}{n}\gamma\right)^n}.$$

This formula can be transformed as follows

$$A_n \geqslant \frac{P[\sigma]^{n-1}}{[\sigma]^n\left(1-\frac{L}{n}\frac{\gamma}{[\sigma]}\right)^n} = \frac{P}{[\sigma]}\frac{1}{\left(1-\frac{L}{n}\frac{\gamma}{[\sigma]}\right)^n}. \tag{3.18}$$

In the case of a rod of uniform strength, when $n \longrightarrow \infty$, we have

$$A_n \geqslant \frac{P}{[\sigma]}e^{\gamma L/[\sigma]}, \tag{3.18 a}$$

where e is the base of the system of natural logarithms.

Example 13. A stepped rod with three portions of prismatic shape and equal length is acted on by a tensile load $P=12$ tons applied at its end. Determine the cross-sectional areas of the portions of the rod, its total elongation and saving of weight as compared with a bar of uniform section if the length of each portion of the rod is $l=50$ m, the allowable stress $[\sigma]=500$ kgf/cm^2, the modulus of elasticity $E=2\times10^6$ kgf/cm^2 and the specific weight $\gamma=$ $=7.8$ gf/cm$^3=0.0078$ kgf/cm^3.

Solution. The cross-sectional area of the first portion, from the bottom, is

$$A_1 = \frac{P}{[\sigma]-\gamma l} = \frac{12{,}000}{500-0.0078\times 5{,}000} = 26 \text{ cm}^2.$$

The weight of the first portion is $W_1 = A_1 l\gamma = 26\times 5{,}000\times 0.0078 = 1{,}010$ kgf. The elongation of the first portion is, by formula (3.17),

$$\Delta l_1 = \frac{Pl}{EA_1} + \frac{W_1 l}{2EA_1} = \frac{12{,}000\times 5{,}000}{2\times 10^6\times 26} + \frac{1{,}010\times 5{,}000}{2\times 2\times 10^6\times 26} = 1.20 \text{ cm}.$$

The cross-sectional area of the second portion is

$$A_2 \geqslant \frac{P[\sigma]}{([\sigma]-\gamma l)^2} = \frac{12{,}000\times 500}{(500-0.0078\times 5{,}000)^2} = 28.2 \text{ cm}^2.$$

The weight of the second portion is

$$W_2 = A_2 l\gamma = 28.2\times 5{,}000\times 0.0078 = 1{,}100 \text{ kgf}.$$

The elongation of the second portion is

$$\Delta l_2 = \frac{(P+W_1)\,l}{EA_2} + \frac{W_2 l}{2EA_2} =$$

$$= \frac{(12{,}000+1{,}010)\,5{,}000}{2\times 10^6\times 28.2} + \frac{1{,}100\times 5{,}000}{2\times 2\times 10^6\times 28.2} = 1.12 \text{ cm}.$$

The cross-sectional area of the third portion is

$$A_3 \geqslant \frac{P[\sigma]^2}{([\sigma]-\gamma l)^3} = \frac{12{,}000\times 500^2}{(500-0.0078\times 5{,}000)^3} = 30.6 \text{ cm}^2.$$

The weight of the third portion is

$$W_3 = A_3\gamma l = 30.6\times 0.0078\times 5{,}000 = 1{,}190 \text{ kgf}.$$

The elongation of the third portion is

$$\Delta l_3 = \frac{(P_1+W_1+W_2)\,l}{EA_3} + \frac{W_3 l}{2EA_3} =$$

$$= \frac{(12{,}000+1{,}010+1{,}100)\,5{,}000}{2\times 10^6\times 30.6} + \frac{1{,}190\times 5{,}000}{2\times 2\times 10^6\times 30.6} = 1 \text{ cm}.$$

The total weight of the rod is

$$W = W_1 + W_2 + W_3 = 1{,}010 + 1{,}100 + 1{,}190 = 3{,}300 \text{ kgf}.$$

The total elongation of the rod is

$$\Delta l = \Delta l_1 + \Delta l_2 + \Delta l_3 = 1.2 + 1.12 + 1 = 3.32 \text{ cm}.$$

If the rod were of uniform section, the required cross-sectional area, as determined by formula (3.11) with the weight of the rod taken

into account, would be

$$A \geqslant \frac{P}{[\sigma]-\gamma l} = \frac{12{,}000}{500-0.0078\times 3\times 5{,}000} = 31.3 \text{ cm}^2$$

The weight of this rod would be

$$W' = AL\gamma = 31.3\times 3\times 5{,}000\times 0.0078 = 3{,}662 \text{ kgf}.$$

Consequently, the saving of weight in the case of the stepped rod is

$$\Delta W = W' - W = 3{,}662 - 3{,}300 = 362 \text{ kgf}$$

or in per cent

$$\frac{362}{3{,}662}\times 100 \cong 10\%.$$

18. Statically Indeterminate Problems in Tension and Compression

In many problems of strength of materials the internal forces induced in rods cannot be determined only by the use of the equations of equilibrium of absolutely rigid bodies. This happens when the number of unknown forces is greater than the number of equilibrium equations that can be set up for the case in hand. Such problems are therefore called *statically indeterminate* problems. They are solved by adding to the equations of equilibrium the lacking number of equations obtained by consideration of elastic deformations. The equations of elastic deformations differ from the equations of equilibrium. They involve, in addition to forces and geometric dimensions, quantities characterizing the elastic properties of a material, i. e., the moduli of elasticity of a material.

In this section are given specific examples illustrating the solution of some statically indeterminate problems in tension and compression.

Example 14. A steel rod of length l and cross-sectional area A cm^2 fixed at both ends is subjected to a force $P=3$ tons (Fig. 37) applied at a section mn a distance $l_1 = 10$ cm from the upper fixed end and $l_2 = 20$ cm from the lower fixed end. Determine the forces in the portions l_1 and l_2 of the rod.

Solution. The force P stretches the upper portion of the rod and compresses the lower portion, therefore both reactions at the fixed ends of the rod are directed upward. Denote the reaction at the upper fixed end by R_1 and at the lower fixed end by R_2. To determine the two reactions statics gives in this case only one equation of equilibrium, $\sum Y = 0$, from which we obtain

$$R_1 + R_2 = P.$$

The second equation is set up by consideration of the deformation of the rod. Since the ends of the rod are fixed, the force P is

obviously distributed between the upper and lower portions of the rod so that the elongation of the upper portion is equal to the contraction of the lower portion. Hence we obtain the second equation

$$\frac{R_1 l_1}{EA} = \frac{R_2 l_2}{EA}$$

or

$$\frac{R_1}{R_2} = \frac{l_2}{l_1},$$

i. e., *the reactions are inversely proportional to the lengths* l_1 and l_2.

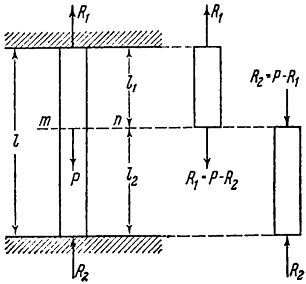


Fig. 37

Solving this equation simultaneously with the equation of statics we find

$$R_1 = P\frac{l_2}{l_1+l_2} \text{ and } R_2 = P\frac{l_1}{l_1+l_2}$$

or, substituting the numerical values, we obtain

$$R_1 = 3\frac{20}{10+20} = 2 \text{ tons and } R_2 = 3\frac{10}{10+20} = 1 \text{ ton.}$$

If $l_1 = l_2$, then $R_1 = R_2 = \frac{1}{2}P$.

Example 15. A steel cylinder (Fig. 38) is inserted into a copper collar. The cylinder and collar are compressed by two absolutely rigid plates with a force P. Determine the stresses induced in the cylinder and collar if $P = 40{,}000$ kgf, the diameter of the steel cylinder $d = 10$ cm, the inner diameter of the collar $d_1 = 11$ cm, the outer diameter $d_2 = 21$ cm, the modulus of elasticity of steel $E_s = 2\times10^6$ kgf/cm², the modulus of elasticity of copper $E_c = 1 \times$ $\times 10^6$ kgf/cm².

Solution. To determine the forces acting on the steel cylinder P_s and on the copper collar P_c, statics gives only one equation

$$P_s + P_c = P. \tag{a}$$

The lacking second equation will be obtained by consideration of deformations. Because of the rigidity of the plates the cylinder and collar undergo equal contractions. The contraction of the steel cylinder is

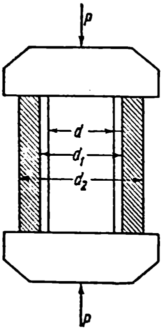


Fig. 38

$$\Delta l_s = \frac{P_s l_s}{E_s A_s}.$$

The contraction of the copper collar is

$$\Delta l_c = \frac{P_c l_c}{E_c A_c}.$$

Since $\Delta l_s = \Delta l_c$ according to the condition of the problem, we have

$$\frac{P_s}{E_s A_s} = \frac{P_c}{E_c A_c}.$$

This deformation equation may be rewritten as

$$\frac{P_s}{P_c} = \frac{E_s A_s}{E_c A_c}. \tag{b}$$

From the equation (b) it is seen that the force P acting on the cylinder and collar is distributed in proportion to their stiffnesses.

Solving the equations (a) and (b) simultaneously, we obtain

$$P_s = P\frac{E_s A_s}{E_s A_s + E_c A_c} \quad \text{and} \quad P_c = P\frac{E_c A_c}{E_c A_c + E_s A_s}.$$

From these expressions it is seen that the forces depend on the stiffnesses of the cylinder and collar.

Dividing these forces by the cross-sectional areas, we find the stresses in the steel cylinder and copper collar, respectively,

$$\sigma_s = P\frac{E_s}{E_s A_s + E_c A_c}, \quad \sigma_c = P\frac{E_c}{E_c A_c + E_s A_s}.$$

Comparing the resulting stresses we note that they are proportional to the moduli of elasticity of the materials and their ratio is independent of the relation between the cross-sectional areas

$$\frac{\sigma_s}{\sigma_c} = \frac{E_s}{E_c}. \tag{c}$$

The same conclusion could be reached in a simpler way: since the strains in the cylinder and collar are equal because of the rigidity of the compression plates, the stresses are proportional to the mo-

duli of elasticity, according to Hooke's law,

$$\frac{\sigma_s}{\sigma_c}=\frac{E_s\varepsilon_s}{E_c\varepsilon_c}=\frac{E_s}{E_c}.$$

Determine now the numerical values of the stresses, substituting for symbols the given data in the corresponding expressions

$$\sigma_s=P\frac{E_s}{E_sA_s+E_cA_c}=$$

$$=40{,}000\frac{2\times 10^6}{2\times 10^6\frac{3.14\times 10^2}{4}+1\times 10^6\frac{3.14\,(21^2-11^2)}{4}}=194\ \text{kgf/cm}^2,$$

$$\sigma_c=\sigma_s\frac{E_c}{E_s}=194\frac{1\times 10^6}{2\times 10^6}=97\ \text{kgf/cm}^2.$$

Example 16. A weight $P=1$ ton (Fig. 39) is suspended by three bars. The cross-sectional area of the vertical steel bar is $A_s=1$ cm^2; either of the side copper bars has a cross-sectional area $A_c=2$ cm^2. Determine the stresses in the bars and find the distance the point of suspension B of the weight will move down if the length of the vertical bar is $l_1=0.5$ m and the angle $\alpha=45°$.

Solution. Denote the tensile force in the vertical bar by X.

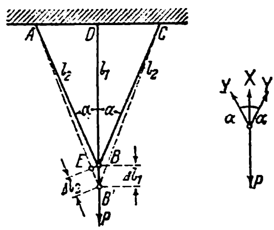


Fig. 39

From the first condition of equilibrium of hinge B (the sum of projections of forces on the horizontal direction is zero) it follows that the forces in the inclined bars are equal. Denote these forces by Y. From the second condition of equilibrium of hinge B (the sum of projections of forces on the vertical direction is zero) we obtain

$$X+2Y\cos\alpha=P. \tag{a}$$

To determine the forces X and Y we obtain the lacking equation from consideration of the deformations of the bars. Assume that the point of suspension of the weight occupies a new position B' after deformation. The positions of the bars after deformation are shown dashed in Fig. 39.

The elongation of the vertical bar is

$$BB'=\Delta l_1=\frac{Xl_1}{E_sA_s}.$$

The elongation of either of the inclined bars, say, the left one, can be determined by dropping a perpendicular from point B upon

the new position of the bar AB'. Since the deformations considered in strength of materials are small, it may be assumed that $AE = AB$, i. e., the perpendicular BE is a circular arc of radius AB and the angle $DB'A = \alpha$. As is seen from triangle $BB'E$, the elongation of the side bar is then

$$EB' = BB' \cos\alpha,$$

$$\Delta l_2 = \Delta l_1 \cos\alpha = \frac{Xl_1}{E_s A_s} \cos\alpha.$$

But the elongation Δl_2 of the side bar can also be determined by Hooke's law

$$\Delta l_2 = \frac{Yl_2}{E_c A_c},$$

consequently,

$$\frac{Yl_2}{E_c A_c} = \frac{Xl_1 \cos\alpha}{E_s A_s}.$$

Since $l_1 = l_2 \cos\alpha$, we have

$$\frac{Y}{E_c A_c} = \frac{X \cos^2\alpha}{E_s A_s}. \tag{b}$$

Solve this deformation equation simultaneously with the equation of statics (a). From the equation (b) we obtain

$$Y = X \cos^2\alpha \frac{E_c A_c}{E_s A_s}. \tag{c}$$

Substitute this value of Y in the equation (a)

$$X + 2X \cos^3\alpha \frac{E_c A_c}{E_s A_s} = P,$$

whence

$$X = \frac{P}{1 + 2\cos^3\alpha \dfrac{E_c A_c}{E_s A_s}}. \tag{d}$$

Substitute the above value of X in the equation (c)

$$Y = \frac{P \cos^2\alpha \dfrac{E_c A_c}{E_s A_s}}{1 + 2\cos^3\alpha \dfrac{E_c A_c}{E_s A_s}}.$$

If all the bars are made of the same material and have the same cross-sectional area, the forces X and Y as determined from the equations (d) and (c) are

$$X = \frac{P}{1 + 2\cos^3\alpha},$$

$$Y = \frac{P \cos^2\alpha}{1 + 2\cos^3\alpha}.$$

Find the numerical values of the forces in the bars

$$\cos\alpha = \cos 45^\circ = \frac{\sqrt{2}}{2}, \quad \cos^2\alpha = 0.5, \quad \cos^3\alpha = 0.353,$$

$$X = \frac{1{,}000}{1 + 2\times 0.353\,\dfrac{1\times 10^6\times 2}{2\times 10^6\times 1}} = \frac{1{,}000}{1.706} = 585 \text{ kgf},$$

$$Y = \frac{1{,}000\times 0.5\,\dfrac{1\times 10^6\times 2}{2\times 10^6\times 1}}{1 + 2\times 0.353\,\dfrac{1\times 10^6\times 2}{2\times 10^6\times 1}} = 293 \text{ kgf}.$$

The stresses in the vertical and side bars are, respectively,

$$\sigma_s = \frac{X}{A_s} = \frac{585}{1} = 585 \text{ kgf/cm}^2, \quad \sigma_c = \frac{Y}{A_c} = \frac{293}{2} \approx 147 \text{ kgf/cm}^2.$$

The point of suspension B of the weight moves down an amount

$$\Delta l_1 = \frac{Xl_1}{E_s A_s} = \frac{585\times 50}{2\times 10^6\times 1} = 0.0146 \text{ cm} = 0.146 \text{ mm}.$$

19. Stresses Due to Temperature Changes

Increasing or decreasing the temperature of a material produces, respectively, its elongation or contraction. Therefore, if the deformation is constrained a member may develop temperature stresses when heated or cooled. Thus, dangerous stresses are likely to occur in a casting during non-uniform cooling. High temperature stresses are present in a turbine disk since the temperatures at the centre of the disk and on its periphery are not the same. Because of the difference between the coefficients of linear expansion of materials temperature stresses also arise in machine or structural parts made of different materials and joined together.

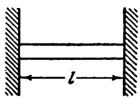


Fig. 40

Consider a simple example illustrating the occurrence of temperature stresses in a statically indeterminate system due to a temperature change. Assume that a steel rod of length l (Fig. 40) is held rigid at its ends between two fixed planes. If one of the ends of the rod were not built in, the rod would be free to elongate or contract because of temperature changes, i. e., no temperature stresses would result from changes in temperature. In the case under consideration, because of the constraint provided by the fixed planes the rod will develop stresses due to changes in temperature.

Assume that the temperature at which the rod is fitted between the planes is t_1 and the coefficient of linear expansion of steel is α. Find the stress in the rod if the temperature changes and becomes equal to t_2.

Denote the change in temperature by $t=t_2-t_1$. The absolute elongation of the rod with one end free would obviously be

$$\Delta l=\alpha(t_2-t_1)\,l=\alpha tl.$$

But the planes prevent the rod from elongating. Consequently, it develops a compressive stress corresponding to the compressive strain $-\varepsilon$.

The compressive strain in the rod is

$$-\varepsilon=\frac{\Delta l}{l}=\frac{\alpha tl}{l}=\alpha t.$$

By Hooke's law, the normal stress is given by

$$\sigma=E\varepsilon=-E\alpha t. \tag{3.19}$$

If $t_1>t_2$, the rod will develop a tensile stress; if $t_1<t_2$, a compressive stress.

Table 6. Coefficients of Linear Expansion of Some Metals

Material	α	Material	α
Aluminium	22.5×10^{-6}	Steel	12×10^{-6}
Bronze	17.5×10^{-6}	Zinc	35.4×10^{-6}
Copper	16.5×10^{-6}	Cast iron	10.4×10^{-6}
Nickel	13×10^{-6}	Elektron	28.5×10^{-6}

To determine the force exerted by the rod on the constraining planes, it is necessary to know the cross-sectional area of the rod. Denoting this force by P and the cross-sectional area of the rod by A, we have

$$P=\sigma A=-E\alpha tA.$$

If the allowable stress for steel is taken as $[\sigma]=1{,}000$ kgf/cm², $E=2\times10^6$ kgf/cm² and the coefficient of linear expansion $\alpha=0.000012$, the stress produced in the rod by a temperature change of $t=50°$ is

$$\sigma=-E\alpha t=-2\times10^6\times0.000012\times50=$$
$$=-1{,}200\ \text{kgf/cm}^2>|1{,}000|\ \text{kgf/cm}^2.$$

As is seen, the stress exceeds the allowable value. From this it follows that stresses due to temperature changes must be taken into account in the design and care should be exerted to avoid these stresses altogether or to make them insignificant. This is achieved by suitably fixing the ends, providing expansion or contraction joints, etc.

Table 6 gives coefficients of linear expansion for some metals.

Example 17. Tramway rails are welded at an ambient temperature of 20° C. What will be the stress in the rails when the temperature rises to 40° if $\alpha = 12\times10^{-6}$ and $E = 2\times10^{6}$ kgf/cm²?

Solution. The temperature change is

$$t = 40^\circ - 20^\circ = 20^\circ.$$

The stress is determined by formula (3.19)

$$\sigma = -E\alpha t = -2\times10^{6}\times12\times10^{-6}\times20 = -480 \text{ kgf/cm}^2.$$

Example 18. A stepped steel rod (Fig. 41) is fixed between two rigid walls at a temperature t_1. Determine the stresses in the two portions of the rod if its temperature is raised to t_2.

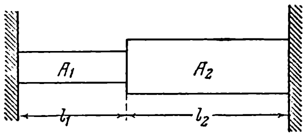


Fig. 41

Solution. In the absence of the walls the absolute elongation of the rod would be

$$\Delta l = \alpha (t_2 - t_1)(l_1 + l_2). \tag{a}$$

The walls prevent the rod from elongating freely; consequently, they compress the rod with a force P such that produces a contraction equal in magnitude to the elongation Δl.

The contraction of the rod produced by the force P is

$$\Delta l = \frac{Pl_1}{EA_1} + \frac{Pl_2}{EA_2}. \tag{b}$$

From the equations (a) and (b) we have

$$\alpha (t_2 - t_1)(l_1 + l_2) = P\left(\frac{l_1}{EA_1} + \frac{l_2}{EA_2}\right),$$

whence

$$P = \frac{\alpha (t_2 - t_1)(l_1 + l_2)}{\frac{l_1}{EA_1} + \frac{l_2}{EA_2}}.$$

The stress in the left-hand portion of the rod is

$$\sigma = \frac{\alpha (t_2 - t_1)(l_1 + l_2)}{\left(\frac{l_1}{EA_1} + \frac{l_2}{EA_2}\right)A_1}.$$

The stress in the right-hand portion of the rod is

$$\sigma = \frac{\alpha (t_2 - t_1)(l_1 + l_2)}{\left(\frac{l_1}{EA_1} + \frac{l_2}{EA_2}\right) A_2} .$$

20. Design of Statically Indeterminate Systems Based on Allowable Loads, and Limit Design

In the preceding sections, in the analysis of both statically determinate and statically indeterminate structures in tension and compression the cross-sectional dimensions were determined from the condition

$$\sigma_{max} \leqslant [\sigma].$$

When this requirement is fulfilled, the maximum stress at the most severely stressed point or points does not exceed the allowable value. In other words, the determination of dimensions was based on the method of allowable stresses.

In the last few years it has become the practice to use, in some cases, the *allowable load* and not the *allowable stress* in the determination of structural dimensions. We shall illustrate the advantages of the new method over the design method based on allowable stresses.

Let it be required to determine the cross-sectional dimensions of a bar made of a mild steel having a yield strength σ_y. If the factor of safety is taken to be k, the cross-sectional area of the bar is determined from the following inequality

$$\frac{P}{A} \leqslant \frac{\sigma_y}{k} \tag{a}$$

or

$$A \geqslant \frac{Pk}{\sigma_y} . \tag{b}$$

We shall now determine the cross-sectional area on the basis of the *allowable load* assuming the same factor of safety k.

In the design based on the allowable load we have

$$P \leqslant P_{al} = \frac{P_{lim}}{k} , \tag{c}$$

where P_{al} is the allowable load and P_{lim} is the limiting load, which is in this case a load producing a stress on the section equal to the yield stress σ_y, i. e.,

$$P_{lim} = A\sigma_y;$$

consequently,

$$P_{al} = \frac{A\sigma_y}{k} .$$

The strength condition (c) becomes

$$P \leqslant \frac{A\sigma_y}{k} \tag{d}$$

or

$$A \geqslant \frac{Pk}{\sigma_y}. \tag{e}$$

Thus, the design based on the allowable load yields in this case the same result as the design based on the allowable stress since the strength conditions [equations (b) and (e)] are identical.

This is also true for more complex statically determinate structures.

The situation is different if the structure is statically indeterminate and made of a ductile material, say, mild steel. Consider a simple example. Assume that a bar of mild steel is fixed at both ends and subjected to a force P as shown in Fig. 42. Determine the cross-sectional area on the basis of the allowable stress and the allowable load with the same factor of safety k. The forces in the upper and lower portions of the bar (see Example 14, p. 60) are, respectively,

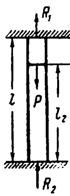


Fig. 42

$$R_1 = P\frac{l_2}{l} \quad \text{and} \quad R_2 = P\frac{l_1}{l}.$$

Let $l_2 > l_1$, then the stresses in the upper portion of the bar are greater than in the lower portion. The cross-sectional area in the design based on the allowable stress is determined from the condition

$$A \geqslant \frac{R_1 k}{\sigma_y} = P\frac{l_2}{l}\frac{k}{\sigma_y}.$$

The stress in the upper portion of the bar has a maximum value σ_y/k. The force R_2 acting in the lower portion is smaller, therefore the induced stress is smaller than σ_y/k and the factor of safety is larger.

Consequently, if the cross-sectional area of the bar is determined on the basis of the allowable stress, the material of the lower portion of the bar will not be fully utilized.

Determine now the cross-sectional area of the bar on the basis of the allowable load. Since the bar is made of a mild steel with a pronounced yield point in the tension (compression) test diagram and a considerable stretching at that point, the stress in the upper portion of the bar, after reaching the yield point, will no longer increase. As the force P increases, the stress increases only in the lower portion of the bar. This will continue up to a value of the force P for which the stress in the lower portion of the bar reaches

the yield point. Only after that does yielding occur throughout the bar with increasing force. In other words, the limiting load in this case is that which produces a stress equal to σ_y in both portions of the bar. After the stress reaches σ_y in the upper portion of the bar, the system becomes as if statically determinate since a part of the limiting force which produces the extension of the upper portion is already known, being $\sigma_y A_1$, where A_1 is the required cross-sectional area of the bar.

Consequently, the other part of the limiting force which produces the compression of the lower portion of the rod is $P_{\text{lim}} - \sigma_y A_1$. When this force induces a stress in the lower portion of the bar which is equal to the yield stress σ_y, the following equality will be satisfied

$$P_{\text{lim}} - \sigma_y A_1 = \sigma_y A_1.$$

Consequently, the limiting load is

$$P_{\text{lim}} = 2\sigma_y A_1.$$

Since the strength condition in the design based on the allowable load is expressed by the inequality

$$P \leqslant P_{al} = \frac{P_{\text{lim}}}{k},$$

we have

$$Pk \leqslant 2\sigma_y A_1,$$

whence the required area is

$$A_1 \geqslant \frac{Pk}{2\sigma_y}.$$

Comparing the area A based on stress with the area A_1 based on load, we see that

$$A_1 < A.$$

The ratio of the areas is

$$\frac{A_1}{A} = \frac{Pk}{2\sigma_y} : \frac{Pk}{\sigma_y} \frac{l_2}{l_1 + l_2} = \frac{l_1 + l_2}{2l_2}.$$

Denote $l_2/l_1 = n$, then

$$\frac{A_1}{A} = \frac{1+n}{2n}.$$

Hence, the ratio of the areas in this case depends on the value of n, i. e., on the point of application of the load P.

Thus, if $l_2/l_1 = 20$, then $A_1/A = 21/40$, i. e., the required cross-sectional area based on load is approximately one-half the area based on stress. Consequently, the quantity of material needed for the fabrication of a part will be reduced by 50 per cent. There will be no saving in material only when $n = 1$. In this case the

designs based on load and stress yield the same result. This is due to the fact that for $n=1$ the stresses are equal throughout the length of the bar.

The design method based on the allowable load more fully utilizes the reserve strength of structures made of ductile materials. By this means saving of material and weight is achieved without any loss in strength.

Example 19. Three bars of mild steel with a yield point stress σ_y and of the same cross-sectional area A support an absolutely rigid rod AB subjected to a force P (Fig. 43). The extreme bars of equal length l are located symmetrically with respect to the middle bar whose length is $\frac{1}{2}l$.

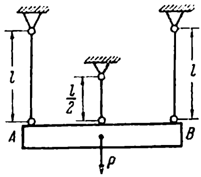


Fig. 43

Compare the allowable forces P as obtained in the designs based on stress and load if the factor of safety k is to be the same in both cases.

Solution. Determine first the allowable force on the basis of the allowable stress. Denote this force by P'_{al}. Since the extreme bars are located symmetrically with respect to the middle bar and the point of application of the force, they will carry equal forces; denote them by P_1. The force carried by the middle bar is denoted by P_2.

From the equilibrium condition we have

$$2P_1+P_2-P'_{al}=0. \tag{a}$$

To determine the independent forces P_1 and P_2 we set up an additional deformation equation. Since the rod is absolutely rigid, the elongations of the bars are the same

$$\frac{P_1 l}{EA}=\frac{P_2\frac{1}{2}l}{EA}$$

or

$$P_1=\frac{1}{2}P_2. \tag{b}$$

From the equations (a) and (b) we find

$$P_1=\frac{1}{4}P'_{al}, \quad P_2=\frac{1}{2}P'_{al}.$$

Since the middle bar carries a larger force, the allowable force P'_{al} is determined from the strength condition for this bar

$$\frac{\frac{1}{2}P'_{al}}{A} \leqslant \frac{\sigma_y}{k} = [\sigma],$$

whence

$$P'_{al} \leqslant \frac{2\sigma_y A}{k}.$$

Determine now the allowable force by the design method based on the allowable load. Denote the limiting value of the force at which the stresses in all bars reach the yield point stress by $P_{\lim}$. As the force increases, the stress in the middle bar reaches the yield point stress sooner than in the extreme bars. When the stress in the middle bar reaches the yield point stress σ_y, this bar will carry a force equal to $\sigma_y A$. Then either of the extreme bars will carry a force of magnitude

$$\frac{1}{2}(P_{\lim} - \sigma_y A).$$

Since the stresses in the extreme bars reach the yield point stress at the limiting value of the force, we have

$$\frac{\frac{1}{2}(P_{\lim} - \sigma_y A)}{A} = \sigma_y.$$

From this we determine the value of $P_{\lim}$

$$P_{\lim} = 3\sigma_y A.$$

The allowable force denoted by P''_{al} is

$$P''_{al} = \frac{P_{\lim}}{k} = \frac{3\sigma_y A}{k}.$$

From comparison of the values of the allowable forces P'_{al} and P''_{al} we conclude that in this case the design method based on the allowable load permits a force 1.5 times that based on the allowable stress.

In 1955 a new structural design method was introduced in the USSR which is obligatory for some design organizations. We shall briefly outline the fundamentals of this method, which differs from the design procedures based on allowable stresses and allowable loads. Three limiting states of structures may be distinguished, depending upon the operating requirements:

(1) design state based on the *load-carrying capacity* of a structure, i. e., the ultimate strength, stability and fatigue of the material;

(2) design state based on the *stiffness* of a structure which implies that the deformations must be kept within the limits laid down by rules of the relevant codes;

(3) design state based on the development of *localized damage* such as the formation and opening of cracks.

For each of the three limiting states, the Structural Design Code establishes appropriate design formulas which serve to guarantee the normal service of an engineering structure.

The evaluation of the first limiting state involves the determination of the so-called load-carrying capacity of a structure or of its individual elements. In the case of central extension the load-carrying capacity of a bar is judged by the magnitude of the force $N_{l.c}$ given by the formula

$$N_{l.c}=AR''km, \tag{I}$$

where A is the cross-sectional area of the bar, R'' is the so-called normative resistance of the material which is often assigned equal to the yield strength, $k\leqslant 1$ is the factor of non-homogeneity of the material which is sometimes taken (for steel) equal to 0.8 to 0.9, $m\leqslant 1$ is the factor of operating conditions which reflects the effect of local weakenings of the section of the bar, the features of its shape (thin-walled section, etc.), the type of service, environmental effects, etc.; since the factor m takes account of the effect of stress concentration as well, its value may be considerably lower than $1/\alpha_{ef}$, where α_{ef} is the effective stress concentration factor (see Sec. 15). The product $R''k$ is sometimes called the design resistance of the material.

In the design, the magnitude of the force $N_{l.c}$ given by formula (I) must be compared with the so-called design force N_d, which is obtained by multiplying the forces acting in the structure by the so-called overloading factors n_1 and n_2

$$N_d=N_1n_1+N_2n_2. \tag{II}$$

Here N_1 and N_2 are the forces due to dead and live loads respectively, the factors n_1 and n_2 are specified by the Code rules assuming that the possible overloading due to a live load (n_2) is generally higher than the overloading due to a dead load (n_1) and ranges from 1.2 to 1.5.

The design formula for the first limiting state in tension is $N_d\leqslant N_{l.c}$ or, after substituting the values of N_d and $N_{l.c}$ from formulas (I) and (II),

$$N_1n_1+N_2n_2\leqslant AR''km. \tag{III}$$

Suppose, for example, that the following data are given: $N_1=2$ tons, $N_2=2.2$ tons, $n_1=1.1$, $n_2=1.5$, $R''=\sigma_y=2.3$ tons/cm², $k=m=0.9$.

From formula (III) we find

$$A = \frac{2\times 1.1 + 2.2\times 1.5}{2.3\times 0.81} = 3 \text{ cm}^2.$$

This is the required cross-sectional area of the bar.

The design formula for the second limiting state is

$$\Delta \leqslant f, \tag{IV}$$

where Δ is the actual displacement in the structure, f is the normative displacement; in the design of tension bars, formula (IV) is rarely used; overloadings are not taken into account here.

The third limiting state is typical in reinforced concrete design.

21. Check Questions

What is the stress concentration factor?

Give the definition of the allowable stress and the factor of safety.

What is taken as the starting limiting stress in the choice of the allowable stress for a brittle material?

What is taken as the starting limiting stress in the choice of the allowable stress for a ductile material?

When is stress concentration neglected in choosing the allowable stress?

What is a rod of uniform strength and where is it encountered in practice?

What is the critical and limiting length of a rod?

What formulas are used to determine stresses and strains in a tension or compression member of uniform section if its own weight is taken into account?

Explain the advantage of a stepped rod over a prism when the effect of gravity is taken into account.

What problems are called statically indeterminate ones?

What additional equations are to be set up for the solution of statically indeterminate problems?

Give an example of the occurrence of temperature stresses.

What is the difference between design procedures based on the allowable load and the allowable stress?

What design limiting states are adopted in the Structural Design Code of 1955?

What are the factors of overloading, non-homogeneity of a material, and operating conditions?

What is the normative resistance?

Explain the design formula (III).

Chapter IV

Combined stresses

22. Stresses on Inclined Sections Under Axial Tension or Compression

In the above discussion of the extension of a rod we determined stresses only on a plane perpendicular to the direction of the acting axial forces.

Determine now the stresses occurring on an inclined section MN of an element of a rod stretched by two opposite forces (Fig. 44a).

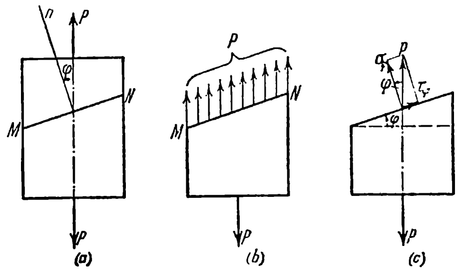


Fig. 44

We agree to measure the positive angle φ of inclination of the section counterclockwise from the direction of the force P to the normal to section MN. Cut the rod through section MN and remove the upper portion (Fig. 44b). To maintain the remaining lower portion in equilibrium, we apply internal elastic forces in the plane of the section. These forces act upward parallel to the force P. Their resultant is equal to P. If the elastic forces are uniformly distributed over the section, the stress on section MN is

$$p = \frac{P}{A_\varphi},$$

where A_φ is the area of section MN.

Denote the area of a normal section by A and the stress on this section by σ; then

$$\sigma = \frac{P}{A}.$$

Since

$$A_\varphi = \frac{A}{\cos\varphi},$$

we have

$$p = \frac{P}{A_\varphi} = \frac{P\cos\varphi}{A} = \sigma\cos\varphi.$$

The stress p on section MN is the total stress on this section. To determine the normal and shearing stresses on section MN we resolve the total stress p into two components as shown in Fig. 44c. The normal stress on the inclined section MN is denoted by σ_φ and the shearing stress by τ_φ

$$\sigma_\varphi = p\cos\varphi = \sigma\cos^2\varphi, \tag{4.1}$$

$$\tau_\varphi = p\sin\varphi = \sigma\sin\varphi\cos\varphi = \frac{\sigma}{2}\sin 2\varphi. \tag{4.2}$$

The shearing stress will be considered positive if its direction coincides with the direction of the normal rotated through 90° clockwise until it coincides with the plane of section MN. In the case considered the stress τ_φ is positive.

Thus, we see that *both normal and shearing stresses occur simultaneously* on inclined sections of a rod under simple tension. From expressions (4.1) and (4.2) it is seen that σ_φ and τ_φ depend on the angle of inclination. Let us see how these stresses vary with the angle φ.

If $\varphi = 0$, i. e., at a transverse section, the normal stress, as is seen from (4.1), attains its maximum value and becomes equal to σ

$$\sigma_{\varphi\,max} = \sigma.$$

The shearing stress is zero on this section, as is seen from expression (4.2),

$$\tau_\varphi = \frac{\sigma}{2}\sin 0 = 0.$$

If $\varphi = 45°$, then

$$\sigma_\varphi = \sigma\cos^2\varphi = \sigma\cos^2 45° = \sigma\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\sigma}{2},$$

$$\tau_\varphi = \frac{\sigma}{2}\sin 2\varphi = \frac{\sigma}{2}\sin 90° = \frac{\sigma}{2}.$$

Consequently, *at sections inclined at an angle of 45° to the direction of tensile forces, the normal and shearing stresses are equal to half the maximum normal stress acting on the transverse section.*

The shearing stresses on planes for which $\varphi = 135^\circ$ and $\varphi = 45^\circ$ attain their maximum absolute values

$$\tau_{\varphi \,\substack{max \\ min}} = \pm \frac{\sigma}{2}. \qquad (4.3)$$

On a longitudinal plane, i. e., for $\varphi = 90^\circ$

$$\sigma_\varphi = \sigma \cos^2 \varphi = \sigma \cos^2 90^\circ = 0,$$

$$\tau_\varphi = \frac{\sigma}{2} \sin 2\varphi = \frac{\sigma}{2} \sin 180^\circ = 0.$$

Hence, *on a longitudinal plane there exist neither normal nor shearing stresses.*

Thus, *in a rod subjected to a longitudinal force there occur both normal and shearing stresses and, as a consequence, the associated extensions and shears.*

This conclusion is of paramount importance in strength of materials.

Many materials, such as mild steel, resist shearing stresses much less than normal stresses. Therefore, in spite of the fact that the maximum shearing stresses in tension or compression are only one half of the maximum normal stresses, they are dangerous and may be the cause of failure of such materials.

Thus, the occurrence of inclined Lüders lines in a tension specimen is accounted for by the effect of shearing stresses. These lines, sometimes visible to the naked eye, reveal the shears produced in the material; their direction depends on the maximum shearing stresses.

On a plane M_1N_1 perpendicular to the plane of section MN (Fig. 45), the normal stress $\sigma_{\varphi+\frac{3\pi}{2}}$ and the shearing stress $\tau_{\varphi+\frac{3\pi}{2}}$ can be determined from formulas (4.1) and (4.2), substituting $\varphi + \frac{3\pi}{2}$ for φ

$$\sigma_{\varphi+\frac{3\pi}{2}} = \sigma \cos^2\left(\varphi + \frac{3\pi}{2}\right) = \sigma \sin^2 \varphi, \quad (4.4)$$

$$\tau_{\varphi+\frac{3\pi}{2}} = \frac{\sigma}{2} \sin 2\left(\varphi + \frac{3\pi}{2}\right) = -\frac{\sigma}{2} \sin 2\varphi. \quad (4.5)$$

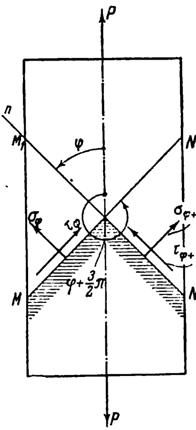


Fig. 45

Adding up the values of the normal stresses on two mutually perpendicular planes (sections MN and M_1N_1), we obtain

$$\sigma_\varphi + \sigma_{\varphi+\frac{3\pi}{2}} = \sigma \cos^2 \varphi + \sigma \sin^2 \varphi = \sigma (\cos^2 \varphi + \sin^2 \varphi) = \sigma, \quad (4.6)$$

i. e., *the algebraic sum of the normal stresses on two mutually perpendicular planes in a tension member is equal to the normal stress* σ *acting on the cross section.*

From comparison of the shearing stresses on two mutually perpendicular planes [formulas (4.2) and (4.5)] we find that

$$\tau_{\varphi} = -\tau_{\varphi + \frac{3\pi}{2}}. \tag{4.7}$$

Consequently, *the shearing stresses on two mutually perpendicular planes are equal in magnitude and opposite in sign (sense).* This important conclusion is known as the *law of equal shearing stresses.* Thus, if shearing stresses τ act on any plane MN of an element, there are numerically equal shearing stresses τ on a perpendicular plane M_1N_1, both being perpendicular to the line of intersection of these planes.

In conclusion we note that all formulas derived in this section for the case of a rod loaded axially in tension are valid for a rod loaded in compression. It should only be remembered that tensile stresses are considered positive and compressive stresses negative.

23. Concept of Principal Stresses

In Sec. 22 we saw that both normal and shearing stresses occur on some planes in a rod under uniaxial tension (compression). Also, in the same section we found that there are sections in a rod on which no shearing stresses occur: these are sections perpendicular to the axis of a stretched (compressed) rod ($\varphi = 0$) and those which are parallel to its axis ($\varphi = 90°$). As we saw, the normal stresses are maximum on the former sections and minimum on the latter sections, being zero in the case considered.

Planes on which no shearing stresses occur are termed principal planes and the normal stresses acting on these planes are called principal stresses.

Principal planes and principal stresses can be defined not only in a rod under axial tension (compression). Under any stress conditions it is possible to pass through each point of a body three mutually perpendicular principal planes, i. e., such planes on which no shearing stresses occur. One plane is associated with the algebraically greatest (maximum) stress σ_1, the second plane with the principal stress σ_2 and the third with the principal stress σ_3, which is the smallest of the three principal stresses. Thus, the numbering of the principal stresses corresponds to the condition

$$\sigma_1 > \sigma_2 > \sigma_3.$$

Thus, if an elementary cube is cut out from a stressed body so that its faces are parallel to the principal planes and the stresses

acting on these planes are $+500$ kgf/cm², -300 kgf/cm², -200 kgf/cm², the principal stresses are numbered as follows

$$\sigma_1 = +500 \text{ kgf/cm}^2, \quad \sigma_2 = -200 \text{ kgf/cm}^2, \quad \sigma_3 = -300 \text{ kgf/cm}^2.$$

If all three principal stresses are different from zero, as in the case above, the state of stress is called a three-dimensional or spatial one.

A two-dimensional or plane state of stress is that in which one of the principal stresses is zero. The case of tension (compression) in two directions falls into this category.

If two principal stresses are zero, the state of stress is called a one-dimensional or linear one. The case of tension (compression) in one direction considered in Sec. 22 and also in Chapters I and II is such a state.

24. Stresses on Inclined Sections Under Tension (Compression) in Two Mutually Perpendicular Directions

Suppose that an element cut out from a prismatic rod is stretched by uniformly distributed stresses in two mutually perpendicular directions (Fig. 46). Since no shearing stresses occur on horizontal and vertical planes in the rod, the normal stresses are principal; therefore, we denote them by σ_1 and σ_2 assuming

$$\sigma_1 > \sigma_2.$$

Determine the stresses on an inclined section perpendicular to the plane of the drawing.

If the rod were stretched only in the horizontal direction, the stresses on section MN would be, according to formulas (4.1) and (4.2),

$$\sigma'_\varphi = \sigma_1 \cos^2 \varphi, \tag{a}$$

$$\tau'_\varphi = \frac{\sigma_1}{2} \sin 2\varphi. \tag{b}$$

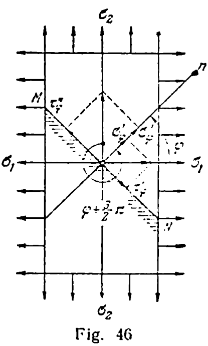


Fig. 46

The stresses on the same section due to extension only in the vertical direction are determined by the same formulas (4.1) and (4.2). Substituting $\varphi + \frac{3}{2}\pi$ for φ on the right-hand sides, we find

$$\sigma''_\varphi = \sigma_2 \sin^2 \varphi, \tag{c}$$

$$\tau''_\varphi = -\frac{\sigma_2}{2} \sin 2\varphi. \tag{d}$$

The total normal stress on section MN is found on the basis of the principle of superposition, i. e., by summation of the expressions (a) and (c)

$$\sigma_\varphi = \sigma'_\varphi + \sigma''_\varphi = \sigma_1 \cos^2 \varphi + \sigma_2 \sin^2 \varphi. \tag{4.8}$$

In a similar way, summing the expressions (b) and (d), we obtain the total shearing stress

$$\tau_\varphi = \tau'_\varphi + \tau''_\varphi = \frac{\sigma_1}{2} \sin 2\varphi - \frac{\sigma_2}{2} \sin 2\varphi$$

or

$$\tau_\varphi = \frac{1}{2} (\sigma_1 - \sigma_2) \sin 2\varphi. \tag{4.9}$$

To determine the maximum and minimum normal stresses we take the first derivative of expression (4.8) with respect to φ and set it equal to zero

$$\frac{d\sigma_\varphi}{d\varphi} = -2\sigma_1 \cos\varphi \sin\varphi + 2\sigma_2 \sin\varphi \cos\varphi = (\sigma_2 - \sigma_1) \sin 2\varphi = 0.$$

This equation is satisfied by two values of the angle φ, namely, $\varphi = 0$ and $\varphi = 90°$. From expression (4.8) it is seen that, if $\sigma_1 > \sigma_2$, σ_{max} occurs when $\varphi = 0°$ and in this case $\sigma_{max} = \sigma_1$; σ_{min} occurs when $\varphi = 90°$, i. e., $\sigma_{min} = \sigma_2$.

Consequently, the principal stresses σ_1 and σ_2 acting on planes on which no shearing stresses exist are the maximum and minimum normal stresses.

As is seen from expression (4.9), the maximum shearing stress occurs when $\sin 2\varphi = 1$, i. e., when $\varphi = 45°$

$$\tau_{max} = \frac{1}{2} (\sigma_1 - \sigma_2). \tag{4.10}$$

If $\sin 2\varphi = -1$, i. e., $\varphi = 135°$, the shearing stress on this plane is numerically equal to τ_{max} but has a minus sign.

Consequently, the maximum shearing stresses are equal to half the difference between the principal stresses and act on planes at $\pm 45°$ to the principal planes.

Example 20. Determine the normal and shearing stresses on a plane inclined at an angle φ to the horizontal face of a rod if the rod is stretched in two mutually perpendicular directions by equal stresses σ.

Solution. By formula (4.8)

$$\sigma_\varphi = \sigma \cos^2\varphi + \sigma \sin^2\varphi = \sigma$$

and by formula (4.9)

$$\tau_\varphi = \frac{1}{2} (\sigma - \sigma) \sin 2\varphi = 0.$$

Consequently, equal normal stresses occur on all sections; shearing stresses are absent.

25. Determination of Principal Stresses

Let us determine the principal stresses in the general case of plane stress. Take an element of a rod whose faces are acted on by uniformly distributed normal stresses σ_x and σ_y and shearing stresses τ (Fig.47*a*). The stresses σ_x and σ_y are not principal stresses since there are also shearing stresses on the planes on which they act. Isolate from the rod an elementary trihedral prism *ABC* with infinitesimal faces which encloses a point *A* (Fig. 47*b*). Determine

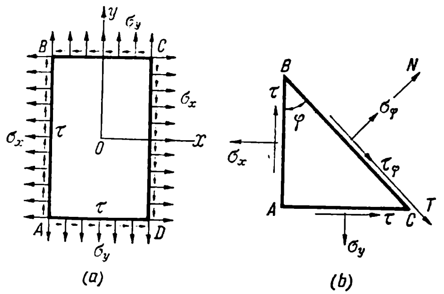


Fig. 47

the stresses σ_φ and τ_φ acting on the inclined plane *BC* from the condition of equilibrium of prism *ABC*.

Denote the area of face *BC* by dA; then the area of face *AC* is obviously $dA \sin\varphi$ and the area of face *AB* is $dA\cos\varphi$. Face *BC* is acted on by the normal force $\sigma_\varphi dA$ and the shearing force $\tau_\varphi dA$. Face *AB* is acted on by the shearing force $\tau\, dA\cos\varphi$ and the normal force $\sigma_x dA\cos\varphi$. Face *AC* is acted on by the shearing force $\tau\, dA\sin\varphi$ and the normal force $\sigma_y dA\sin\varphi$. The required stresses σ_φ and τ_φ are found by projecting all forces acting on the isolated prism on the directions of the stresses σ_φ and τ_φ and equating the sums of the projections of these forces to zero.

Projecting the forces on the *N* axis, we obtain

$$\Sigma N = \sigma_\varphi dA - \sigma_x dA\cos^2\varphi + \tau dA\sin\varphi\cos\varphi + \\ + \tau\, dA\cos\varphi\sin\varphi - \sigma_y dA\sin^2\varphi = 0;$$

on the *T* axis

$$\Sigma T = \tau_\varphi dA - \sigma_x dA\sin\varphi\cos\varphi + \tau\, dA\sin\varphi\sin\varphi - \\ - \tau\, dA\cos\varphi\cos\varphi + \sigma_y dA\sin\varphi\cos\varphi = 0.$$

Dividing through by dA and taking into account that

$$\sin^2\varphi = \frac{1}{2}(1 - \cos 2\varphi),\ 2\sin\varphi\cos\varphi = \sin 2\varphi,\ \cos^2\varphi - \sin^2\varphi = \cos 2\varphi,$$

we obtain the following expressions for the normal and shearing stresses

$$\sigma_\varphi = \sigma_x \cos^2\varphi + \sigma_y \sin^2\varphi - \tau \sin 2\varphi, \tag{4.11}$$

$$\tau_\varphi = \frac{1}{2}\sigma_x \sin 2\varphi - \frac{1}{2}\sigma_y \sin 2\varphi + \tau\cos 2\varphi. \tag{4.12}$$

Since

$$\sin^2\varphi = \frac{1-\cos 2\varphi}{2}, \qquad \cos^2\varphi = \frac{1+\cos 2\varphi}{2},$$

expressions (4.11) and (4.12) can be rewritten as

$$\sigma_\varphi = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\varphi - \tau\sin 2\varphi, \tag{4.11a}$$

$$\tau_\varphi = \frac{\sigma_x-\sigma_y}{2}\sin 2\varphi + \tau\cos 2\varphi. \tag{4.12a}$$

These stresses depend on the angle of inclination of a plane. Find the principal stresses; they act on planes on which there are no shearing stresses, therefore the position of principal planes can easily be determined from Eq. (4.12a) by equating τ_φ to zero. Then

$$\tan 2\varphi = -\frac{2\tau}{\sigma_x-\sigma_y}. \tag{4.13}$$

From this equation we obtain two values for the angle φ differing by 90°. One of the values of φ corresponds to the maximum value of σ_φ and the other, to the minimum. This can easily be verified since the first derivative of σ_φ with respect to φ, i. e.,

$$\frac{d\sigma_\varphi}{d\varphi} = -\frac{\sigma_x-\sigma_y}{2}2\sin 2\varphi - 2\tau\cos 2\varphi$$

becomes zero if we substitute the value of the angle given by formula (4.13).

Thus, the maximum and minimum stresses, i. e., the principal stresses, act on two mutually perpendicular principal planes for each point of a rod. To determine the principal stresses we recall that the trigonometric functions appearing in Eq. (4.11) can be represented as

$$\sin 2\varphi = \pm\frac{\tan 2\varphi}{\sqrt{1+\tan^2 2\varphi}}, \qquad \cos 2\varphi = \pm\frac{1}{\sqrt{1+\tan^2 2\varphi}}.$$

Equation (4.11a) can now be rewritten as

$$\sigma_\varphi = \frac{\sigma_x+\sigma_y}{2} \pm \frac{\sigma_x-\sigma_y}{2}\,\frac{1}{\sqrt{1+\tan^2 2\varphi}} \mp \tau\frac{\tan 2\varphi}{\sqrt{1+\tan^2 2\varphi}}.$$

To calculate the principal stresses we substitute the value of $\tan 2\varphi$ [see formula (4.13)] in the last expression

$$\sigma_\varphi = \frac{\sigma_x+\sigma_y}{2} \pm \frac{\sigma_x-\sigma_y}{2}\,\frac{1}{\sqrt{1+\dfrac{4\tau^2}{(\sigma_x-\sigma_y)^2}}} \pm \tau\,\frac{\dfrac{2\tau}{(\sigma_x-\sigma_y)}}{\sqrt{1+\dfrac{4\tau^2}{(\sigma_x-\sigma_y)^2}}}\,.$$

After transformations we obtain

$$\sigma_\varphi = \frac{\sigma_x+\sigma_y}{2} \pm \frac{1}{2}\sqrt{(\sigma_x-\sigma_y)+4\tau^2}\,.$$

Thus, the principal stresses (the maximum and minimum normal stresses) are, respectively,

$$\left.\begin{aligned} \sigma_{\max} &= \frac{\sigma_x+\sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2}\,, \\ \sigma_{\min} &= \frac{\sigma_x+\sigma_y}{2} - \frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2}\,. \end{aligned}\right\} \qquad (4.14)$$

Adding the two principal stresses together, we obtain

$$\sigma_{\max}+\sigma_{\min} = \sigma_x+\sigma_y.$$

From this it follows that in the general case of plane stress, as in the case of simple tension [see formula (4.6)], the sum of the normal stresses acting on two mutually perpendicular planes is constant and equal to the sum of the principal stresses.

In Sec. 24 we found that the maximum shearing stresses are equal to half the difference between the principal stresses and act on planes inclined at 45° to the principal planes.

Consequently, the maximum shearing stress is

$$\tau_{\max} = \frac{\sigma_{\max}-\sigma_{\min}}{2} = \frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2}. \qquad (4.15)$$

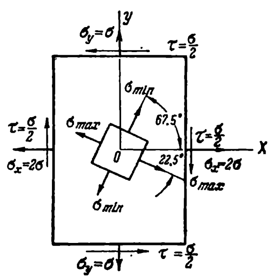


Fig. 48

This result could be obtained from Eq. (4.12a) in the same way as the value of $\sigma_{\max}$ was obtained from Eq. (4.11a).

Example 21. Determine the principal stresses and the position of the principal planes for an element (Fig. 48) subjected to the

stresses

$$\sigma_x = 2\sigma, \quad \sigma_y = \sigma, \quad \tau = \frac{\sigma}{2}.$$

By formulas (4.14) we determine the principal stresses

$$\sigma_{\max} = \frac{2\sigma+\sigma}{2} + \frac{1}{2}\sqrt{(2\sigma-\sigma)^2 + 4\frac{\sigma^2}{2}} = 2.205\sigma,$$

$$\sigma_{\min} = \frac{2\sigma+\sigma}{2} - \frac{1}{2}\sqrt{(2\sigma-\sigma)^2 + 4\frac{\sigma^2}{2}} = 0.795\sigma.$$

The orientation of the principal planes is determined using formula (4.13)

$$\tan 2\varphi = -\frac{\sigma}{\sigma} = -1,$$

$$2\varphi_1 = 135^\circ, \quad \varphi_1 = 67.5^\circ, \quad 2\varphi_2 = -45^\circ, \quad \varphi_2 = -22.5^\circ.$$

26. Strains Under Tension or Compression in Two Mutually Perpendicular Directions. Strain Energy

Suppose that a rod of rectangular section (Fig. 49) is stretched in two mutually perpendicular directions x and y by stresses σ_1 and σ_2. Determine the unit elongations produced in the rod in the x, y and z directions.

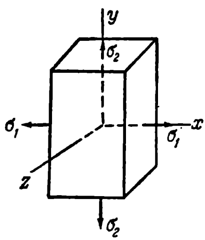


Fig. 49

If the rod were acted on by only the tensile stress σ_1, the unit elongation in the direction of extension would be $_1/E$, according to Hooke's law (2.3), and in the y and z directions the rod would undergo a unit contraction $-\mu\sigma_1/E$.

Likewise, if the tensile stress σ_2 were acting alone, the unit elongation in the direction of extension would be σ_2/E, and in the x and z directions the rod would undergo the unit contraction $-\mu\sigma_2/E$.

Consequently, when the stresses σ_1 and σ_2 are acting simultaneously, the strains in the x, y and z directions are, respectively,

$$\varepsilon_1 = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E},$$

$$\varepsilon_2 = \frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E},$$

$$\varepsilon_3 = -\mu\frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}.$$

In the general case of a three-dimensional state of stress the strains ε_1, ε_2 and ε_3 are, respectively,

$$\left.\begin{aligned}\varepsilon_1 &= \frac{\sigma_1}{E} - \mu\left(\frac{\sigma_2}{E} + \frac{\sigma_3}{E}\right),\\ \varepsilon_2 &= \frac{\sigma_2}{E} - \mu\left(\frac{\sigma_1}{E} + \frac{\sigma_3}{E}\right),\\ \varepsilon_3 &= \frac{\sigma_3}{E} - \mu\left(\frac{\sigma_1}{E} + \frac{\sigma_2}{E}\right).\end{aligned}\right\} \tag{4.16}$$

Special cases.

(1) Simple tension: $\sigma_1 = \sigma$, $\sigma_2 = 0$, $\sigma_3 = 0$; by formulas (4.16) we then obtain

$$\varepsilon_1 = \frac{\sigma}{E}, \quad \varepsilon_2 = -\mu\frac{\sigma}{E}, \quad \varepsilon_3 = -\mu\frac{\sigma}{E}.$$

(2) Tension in two mutually perpendicular directions if $\sigma_1 = \sigma_2 = \sigma$, $\sigma_3 = 0$; from formulas (4.16) we then obtain

$$\left.\begin{aligned}\varepsilon_1 = \varepsilon_2 &= \frac{\sigma}{E}(1-\mu),\\ \varepsilon_3 &= -\frac{2\sigma}{E}\mu.\end{aligned}\right\} \tag{4.17}$$

If the state of stress is two-dimensional, i. e., $\sigma_3 = 0$, and the strains ε_1 and ε_2 are known, formulas (4.16) provide a simple means for determining the stresses σ_1 and σ_2

$$\left.\begin{aligned}\sigma_1 &= \frac{E}{1-\mu^2}(\varepsilon_1 + \mu\varepsilon_2),\\ \sigma_2 &= \frac{E}{1-\mu^2}(\varepsilon_2 + \mu\varepsilon_1).\end{aligned}\right\} \tag{4.18}$$

These formulas are often used in the experimental determination of stresses by measuring ε_1 and ε_2.

Determine the change in volume of a cube of side equal to unity if it is stretched in three mutually perpendicular directions x, y and z. If before deformation the volume of the cube was unity ($v_0 = 1$), then after deformation, because of the change in length of the edges, its volume is

$$v_1 = (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3);$$

neglecting the products of small strains, we obtain

$$v_1 = 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3.$$

The increase in volume is

$$v_1 - v_0 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3.$$

Substituting the values of ε_1, ε_2 and ε_3 from formulas (4.16), we obtain

$$v_1 - v_0 = \left[\frac{\sigma_1}{E} - \mu\left(\frac{\sigma_2}{E} + \frac{\sigma_3}{E}\right)\right] + \left[\frac{\sigma_2}{E} - \mu\left(\frac{\sigma_1}{E} + \frac{\sigma_3}{E}\right)\right] + $$
$$+ \left[\frac{\sigma_3}{E} - \mu\left(\frac{\sigma_1}{E} + \frac{\sigma_2}{E}\right)\right]$$

or

$$v_1 - v_0 = \frac{\sigma_1 + \sigma_2 + \sigma_3}{E}(1 - 2\mu). \tag{4.19}$$

From this equality it is seen that, when $\mu = 0.5$, no change in volume of the element occurs under any stress conditions.

Let us now find an expression for the strain energy per unit volume due to tension or compression in two directions. The strain energy per unit volume due to tension (compression) in one direction is expressed by the formula

$$u = \frac{\sigma^2}{2E} = \frac{\sigma\varepsilon}{2}. \tag{2.9}$$

Under plane-stress conditions, if the faces of a cube of side equal to unity are oriented so as to coincide with the planes on which the stresses σ_1 and σ_2 are acting, we obtain

$$u = \frac{\sigma_1\varepsilon_1}{2} + \frac{\sigma_2\varepsilon_2}{2}. \tag{4.20}$$

Substituting the values of ε_1 and ε_2 from formulas (4.16), with $\sigma_3 = 0$, we have

$$u = \frac{\sigma_1}{2}\left(\frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}\right) + \frac{\sigma_2}{2}\left(\frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E}\right)$$

or

$$u = \frac{1}{2E}(\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2). \tag{4.21}$$

This is an expression for the strain energy per unit volume under plane stress.

Example 22. Determine the stresses σ_1 and σ_2 acting in two mutually perpendicular directions if the measured strains in these directions are, respectively, $\varepsilon_1 = 0.00075$, $\varepsilon_2 = -0.00065$. It is also known that

$$E = 2\times 10^6 \text{ kgf/cm}^2, \quad \mu = 0.3.$$

Solution. The stresses σ_1 and σ_2 are calculated by formulas (4.18)

$$\sigma_1 = \frac{2\times 10^6}{1-0.3^2}(0.00075 - 0.3\times 0.00065) = 1{,}220 \text{ kgf/cm}^2,$$

$$\sigma_2 = \frac{2\times 10^6}{1-0.3^2}(-0.00065 + 0.3\times 0.00075) = -935 \text{ kgf/cm}^2.$$

Example 23. Determine the strains ε_1 and ε_2 in a rod if the tensile stresses are $\sigma_1 = 1{,}000$ kgf/cm² and $\sigma_2 = 500$ kgf/cm², the modulus of elasticity $E = 2\times 10^6$ kgf/cm², Poisson's ratio $\mu = 0.3$.

Solution. On the basis of formulas (4.16) we find

$$\varepsilon_1 = \frac{1{,}000}{2\times 10^6} - 0.3\,\frac{500}{2\times 10^6} = 0.0005 - 0.000075 = 0.000425,$$

$$\varepsilon_2 = \frac{500}{2\times 10^6} - 0.3\,\frac{1{,}000}{2\times 10^6} = 0.00025 - 0.00015 = 0.0001.$$

Example 24. Determine the strains ε_1 and ε_2 in a rod if the tensile stress is $\sigma_1 = 600$ kgf/cm² and the compressive stress is $\sigma_2 = -750$ kgf/cm² (Fig. 50). The modulus of elasticity is $E = 2.2\times 10^6$ kgf/cm², Poisson's ratio $\mu = 0.3$.

Solution. On the basis of formulas (4.16) we have

$$\varepsilon_1 = \frac{600}{2.2\times 10^6} - 0.3\,\frac{-750}{2.2\times 10^6} = 0.000375,$$

$$\varepsilon_2 = \frac{-750}{2.2\times 10^6} - 0.3\,\frac{600}{2.2\times 10^6} = -0.000423.$$

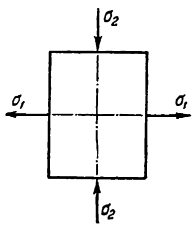


Fig. 50

Example 25. Determine the ratio of the tensile stresses σ_1 and σ_2 (Fig. 49) acting on a rod in two mutually perpendicular directions if the strain ε_1 is to be zero.

Solution. Substituting zero for ε_1 in formula (4.16), we obtain

$$0 = \frac{\sigma_1}{E} - \mu\,\frac{\sigma_2}{E},$$

whence

$$\frac{\sigma_1}{\sigma_2} = \mu.$$

27. Strength Theories

In the analysis of stresses on inclined sections of an axially loaded rod (Sec. 22) we saw that there were both normal and shearing stresses and the associated linear and angular deformations at these sections. Therefore, even in the simplest state of stress, such as the extension of a rod in one direction, the cause of failure of the material may be normal or shearing stresses reaching certain limiting values for the material at hand. In the following discussion, by failure of the material we agree to understand the onset of yielding for ductile materials and fracture for brittle materials. As stated above (Sec. 9) the division of materials into brittle and ductile ones is a matter of convenience. A material behaving as ductile under simple tension may fracture as a brittle

material without any appreciable permanent deformation in the case of all-round tension, and conversely, a material which is brittle in simple tension may behave as ductile under other stress conditions. Therefore, it would be more correct to speak not of a brittle or ductile material but of a brittle or ductile state of the material.

For a brittle material, such as cast iron, the resistance to separation of particles is less than the resistance to slip. Therefore, the cohesion between its individual particles is ruptured prior to occurrence of perceptible permanent deformation, and failure occurs by direct separation. For a ductile material, such as mild steel, the resistance to slip is initially less than the resistance to separation. Therefore, the elements of a crystal lattice slide along crystallographic planes in the material and in consequence permanent set occurs. With the appearance of first permanent deformations the resistance to slip begins to grow. The final rupture of the material is accompanied by considerable plastic deformation.

Thus, the strength of materials which are in a brittle state is characterized by the resistance to separation of particles, and the strength of ductile materials is characterized by the resistance to development of permanent deformation, i. e., the resistance to slip.

For a rod subjected to uniaxial stress, the actual cause of failure of the material is not of great practical significance since the allowable stresses can always be determined from the results of direct testing of the material.

It is different with combined stresses, when a rod, say, is subjected to tension in two mutually perpendicular directions. In such cases the experimental determination of quantities characterizing the conditions of failure of the material and the detection of causes of failure involve considerable difficulties.

In order to predict the onset of failure of the material under combined stresses from the yield strength or the ultimate strength as obtained in simple axial tests, it is necessary to know the actual cause of failure of the material. Up to now several assumptions have been advanced on the basis of theoretical and experimental investigations regarding the cause of failure of materials. These assumptions are known as strength theories. The object of strength theories is to assess the possibility of failure of the material under combined stresses on the basis of material characteristics obtained from axial tests in tension or compression.

The maximum normal stress theory. At the basis of this strength theory proposed by Galileo Galilei is the assumption that the material fails due to the maximum normal stresses. In other words, no matter how complex the state of stress, failure of the material occurs when the normal stress in any one direction reaches the value at which failure occurs in the case of simple tension or compression. Suppose

that a rod (Fig. 51*a*) is stretched in one direction and fails at a normal stress σ. Then, according to this theory, the rod subjected to tension in three mutually perpendicular directions (Fig. 51*b*) with $\sigma_1 > \sigma_2 > \sigma_3$ begins to fail when the maximum stress σ_1 reaches the value of σ. The stresses σ_2 and σ_3, which are smaller than σ_1, are disregarded in this strength theory.

For materials of equal strength in tension and compression, the strength condition is

$$\sigma_1 \leqslant [\sigma].$$

For materials for which the allowable compressive stress $[\sigma_c]$ is not equal to the allowable tensile stress $[\sigma_t]$, the strength must be verified both in tension and compression.

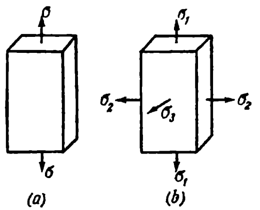


Fig. 51

If, for example, $\sigma_1 > 0$, $\sigma_2 > 0$ and $\sigma_3 < 0$, the strength conditions are

$$\sigma_1 \leqslant [\sigma_t], \quad |\sigma_3| \leqslant [\sigma_c]. \tag{4.22}$$

The first strength theory was proposed before any other; at the time of its development most structural materials were brittle materials (cast iron, stone, etc.). The observation of their failure suggested the idea to the originators of this theory that the cause of failure of all materials is the maximum normal stress. This theory gives fairly satisfactory results only in the analysis of parts made of very brittle materials. The beginning of failure of ductile materials, i. e., the onset of yielding due to high shearing stresses, cannot be explained by this theory. Moreover, one of the serious objections to the first strength theory is that, as experiments show, the stresses sustained by a cube under all-round compression are many times those sustained under simple compression.

The maximum linear strain theory. At the basis of this strength theory is the assumption that no matter how complex the state of stress, the material fails when the maximum tensile or compressive strain in any one direction reaches the value at which failure occurs under simple tension or compression. This theory was originally outlined in general terms in the nineties of the XVII century. In the XIX century it was developed by B. de Saint-Venant.

If a rod is subjected to combined stresses and the principal stresses σ_1, σ_2 and σ_3 are known, the maximum linear strain occurs in the direction of one of the principal stresses. In the case of combined stresses the linear strains in the directions of the princi-

pal stresses are defined by the formulas

$$\left.\begin{aligned}\varepsilon_1&=\frac{\sigma_1}{E}-\frac{\mu}{E}(\sigma_2+\sigma_3),\\ \varepsilon_2&=\frac{\sigma_2}{E}-\frac{\mu}{E}(\sigma_3+\sigma_1),\\ \varepsilon_3&=\frac{\sigma_3}{E}-\frac{\mu}{E}(\sigma_1+\sigma_2).\end{aligned}\right\}\tag{4.16}$$

It will be recalled that in using these formulas the proper signs for the stresses σ_1, σ_2 and σ_3 should be observed. Depending on the relation between the values of σ_1, σ_2 and σ_3, it is always possible to determine the most dangerous strain for a given material from formulas (4.16). According to the second strength theory the most dangerous strain, which will be denoted by ε_{max}, must not be larger than the allowable strain $[\varepsilon]$ in simple tension or compression, i. e.,

$$\varepsilon_{max}\leqslant[\varepsilon].\tag{4.23}$$

The allowable strain is determined from the known formula

$$[\varepsilon]=\frac{[\sigma]}{E},\tag{4.24}$$

where $[\sigma]$ is the allowable stress in tension or compression.

Suppose that the most dangerous tensile strain as determined by formulas (4.16) is the strain ε_1, i. e., $\varepsilon_{max}=\varepsilon_1$. Substituting it in (4.23), we obtain then

$$\varepsilon_1\leqslant[\varepsilon].\tag{4.25}$$

In order to avoid calculating strains, it is more convenient to express the strength condition (4.25) in terms of stresses by substituting the expressions for ε_1 and $[\varepsilon]$ in (4.25)

$$\sigma_1-\mu(\sigma_2+\sigma_3)\leqslant[\sigma].\tag{4.26}$$

If the dangerous tensile strain is ε_2, the strength condition becomes

$$\sigma_2-\mu(\sigma_3+\sigma_1)\leqslant[\sigma].\tag{4.27}$$

Similarly it is possible to set up the strength condition based on the expression for ε_3

$$\sigma_3-\mu(\sigma_1+\sigma_2)\leqslant[\sigma].\tag{4.27'}$$

The left-hand sides of (4.26) and (4.27) represent some stresses which are called *equivalent stresses* and denoted by σ_{eq}

$$\left.\begin{aligned}\sigma_{eq}&=\sigma_1-\mu(\sigma_2+\sigma_3)\leqslant[\sigma],\\ \sigma_{eq}&=\sigma_2-\mu(\sigma_3+\sigma_1)\leqslant[\sigma],\\ \sigma_{eq}&=\sigma_3-\mu(\sigma_1+\sigma_2)\leqslant[\sigma].\end{aligned}\right\}\tag{4.28}$$

Thus, the use of this strength theory involves the determination of the maximum equivalent stress by formulas (4.28) which must not exceed the allowable stress. The concept of an equivalent stress which does not actually exist in the rod is introduced only to avoid calculating strains. The equivalent stress is equal to the stress which would result in an axially loaded rod if the strain produced in it were equal to the maximum strain in the rod under combined stresses.

The second strength theory, even though it takes account of all three principal stresses, is not well supported by experiments and sometimes conflicts with them. Thus, according to this theory a rod subjected to tension in two mutually perpendicular directions must withstand a larger load than when stretched in one direction. Experimental observations do not support this conclusion.

The maximum shearing stress theory. At the basis of this strength theory is the assumption that the chief cause of failure of a material by yielding is the maximum shearing stress. This theory was proposed by C. A. Coulomb in the eighties of the XVIII century. According to this theory, no matter how complex the state of stress, yielding occurs when the maximum shearing stress reaches the value at which failure (yielding) occurs in the case of simple tension.

In Sec. 24 we derived a formula for determining the maximum shearing stress when a rod is subjected to stresses σ_1 and σ_2 in two mutually perpendicular directions.

In this case the maximum shearing stress is

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}. \tag{4.10}$$

If a rod is subjected to stresses σ_1, σ_2 and σ_3 in three mutually perpendicular directions, the maximum shearing stresses are determined from the formulas

$$\tau_1 = \frac{\sigma_2 - \sigma_3}{2}, \quad \tau_2 = \frac{\sigma_1 - \sigma_3}{2}, \quad \tau_3 = \frac{\sigma_1 - \sigma_2}{2}. \tag{4.29}$$

The stresses σ_1, σ_2 and σ_3 are substituted in these formulas with their proper signs. Let $\sigma_1 > \sigma_2 > \sigma_3$; the maximum shearing stress is then

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}. \tag{4.30}$$

If $\sigma_1 > \sigma_2 > 0$ and $\sigma_3 = 0$, the maximum shearing stress is

$$\tau_{max} = \frac{\sigma_1}{2}.$$

To satisfy the strength condition according to the third strength theory, the maximum shearing stress must not exceed the maxi-

mum shearing stress allowed in simple tension

$$\tau_{max} \leqslant [\tau].$$

If the allowable normal stress in simple tension is taken as $[\sigma]$, the allowable shearing stress is, according to formula (4.3),

$$[\tau] = \frac{[\sigma]}{2}.$$

Consequently, the strength condition is

$$\tau_{max} \leqslant \frac{[\sigma]}{2} \tag{4.31}$$

or, taking into account (4.30), we obtain

$$\sigma_1 - \sigma_3 \leqslant [\sigma]. \tag{4.32}$$

The stress $\sigma_1 - \sigma_3$ may be called an equivalent stress, therefore the strength condition (4.32) is written as

$$\sigma_{eq} = \sigma_1 - \sigma_3 \leqslant [\sigma]. \tag{4.33}$$

Thus, according to this strength theory failure occurs when the difference between the maximum and minimum normal stresses reaches a limiting value for a given material.

This strength theory is in fairly good agreement with experimental results for ductile materials, such as mild steel, etc. For materials such as cast iron for which $[\sigma_t] \neq [\sigma_c]$, a correction is introduced in the strength condition (4.33).

A strength theory, which generalizes the third strength theory to cases where $[\sigma_t] \neq [\sigma_c]$, is known as *Mohr's strength theory.*

For the case when $\sigma_3 < 0$ and $[\sigma_t] \neq [\sigma_c]$ the strength condition is, by Mohr's theory,

$$\sigma_{eq} = \sigma_1 - \sigma_3 \frac{[\sigma_t]}{[\sigma_c]} \leqslant [\sigma_t]. \tag{4.34}$$

For particular cases when $[\sigma_t] = [\sigma_c]$ this strength theory is identical with the third strength theory. The maximum shearing stress theory and Mohr's theory are in better agreement with experiment than the first two theories. However, they, too, cannot be recognized as perfect.

The failure of a rod stretched in three mutually perpendicular directions with equal stresses, i. e., when $\sigma_1 = \sigma_2 = \sigma_3 > 0$, cannot be accounted for by shearing stresses, which are zero in this case ($\tau_1 = \tau_2 = \tau_3 = 0$).

The distortion energy theory. The energy theory, which has found wide application for ductile materials, is based on the assumption that, no matter what the state of stress, failure of a material occurs when the distortion energy per unit volume reaches a definite value.

In Sec. 26 we derived the expression for the strain energy per unit volume (4.21) in the case of plane stress

$$u=\frac{1}{2E}(\sigma_1^2+\sigma_2^2-2\mu\sigma_1\sigma_2). \qquad (4.21)$$

This strain energy may be thought of as consisting of two parts: one part is absorbed in changing the volume of a cube and the other, in changing its shape. In Sec. 26 it was shown that no change in volume of a body occurs if $\mu=0.5$. On this basis, setting Poisson's ratio μ equal to 0.5 in expression (4.21), we obtain the strain energy which has no effect in changing the volume, i. e., the strain energy of changing the shape only

$$u_s=\frac{1}{2E}(\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2).$$

In the case of simple tension, when the stress becomes equal to the allowable value, the strain energy is $\frac{1}{2E}[\sigma]^2$.

Consequently, the strength condition for plane stress is, according to the fourth strength theory,

$$\frac{1}{2E}(\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2)\leqslant\frac{1}{2E}[\sigma]^2$$

or

$$\sqrt{\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2}\leqslant[\sigma]. \qquad (4.35)$$

For a three-dimensional state of stress

$$\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_1\sigma_3)}\leqslant[\sigma]. \qquad (4.36)$$

The left-hand sides of the strength conditions (4.35) and (4.36) represent equivalent stresses; denoting these stresses by σ_{eq}, as before, we rewrite the strength conditions as

$$\sigma_{eq}=\sqrt{\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2}\leqslant[\sigma], \qquad (4.37)$$

$$\sigma_{eq}=\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_1\sigma_3)}\leqslant[\sigma]. \qquad (4.38)$$

This theory, suggested at the beginning of our century, gives results which coincide closely with the third strength theory [formula (4.33)], and is suitable for ductile materials.

In spite of its great practical importance, the question of failure of materials is as yet inadequately worked out. Possibilities are studied of generalizing and extending Mohr's strength theory.

The problem of developing a combined strength theory which would take account of the nature of failure of a material and its relation to material properties and the type of state of stress is still to be tackled.

28. Design of Thin-Walled Vessels

Thin-walled vessels are vessels whose wall thickness is small compared with the vessel dimensions and the radii of curvature of the walls are not less than 20 times their thickness.

In the design of thin-walled vessels it is assumed that thin walls do not resist bending and that they develop only tensile or compressive stresses which are uniformly distributed across the wall thickness. Under these assumptions the design of tanks, steam boilers, engine cylinders, etc. yields wholly satisfactory results.

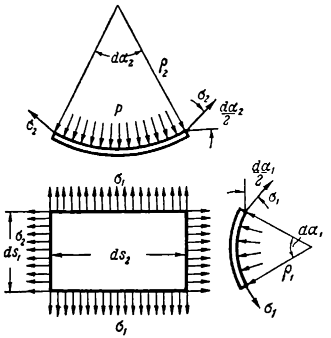


Fig. 52

We shall derive a formula for the design of thin-walled vessels in the shape of a body of revolution under internal pressure p.

Denote the wall thickness by δ, the gauge pressure by p, the radius of curvature corresponding to a longitudinal section (through a meridian) by ρ_1, the radius of curvature corresponding to a transverse section (through a latitude) by ρ_2.

Cut out an infinitesimal element from the wall by two meridional and two normal sections (Fig. 52). At these sections there act stresses σ_1 and σ_2 in two mutually perpendicular directions, i. e., the cut-out element is in a state of plane stress. This element is acted on by the following forces: at a section whose length is denoted by ds_2

$$dN_1 = \sigma_1 \delta \, ds_2,$$

at a section whose length is denoted by ds_1

$$dN_2 = \sigma_2 \delta \, ds_1.$$

These forces must balance the force which is applied to the surface of the element and exerted by the pressure in the vessel, i. e., the force

$$dP = p\, ds_1 ds_2.$$

Set up the equation of equilibrium for the forces acting on the cut-out element. For this purpose we project the forces dN_1, dN_2 and dP on the direction of the normal to the surface of the element

$$2dN_1 \sin\frac{d\alpha_1}{2} + 2dN_2 \sin\frac{d\alpha_2}{2} - dP = 0.$$

The angles $d\alpha_1$ and $d\alpha_2$ are infinitesimally small, therefore it may approximately be taken that

$$\sin\frac{d\alpha_1}{2} = \frac{d\alpha_1}{2}, \quad \sin\frac{d\alpha_2}{2} = \frac{d\alpha_2}{2}.$$

Consequently, the equilibrium equation can be rewritten as

$$dN_1\, d\alpha_1 + dN_2\, d\alpha_2 = dP.$$

The infinitesimal angles between the sections are

$$d\alpha_1 = \frac{ds_1}{\rho_1}, \qquad d\alpha_2 = \frac{ds_2}{\rho_2}.$$

Substituting the values of the forces and angles in the equilibrium equation, we obtain

$$\sigma_1 \delta\, ds_2 \frac{ds_1}{\rho_1} + \sigma_2 \delta\, ds_1 \frac{ds_2}{\rho_2} = p\, ds_1\, ds_2.$$

Dividing through by the product $ds_1\, ds_2\, \delta$, we obtain Laplace's equation

$$\frac{\sigma_1}{\rho_1} + \frac{\sigma_2}{\rho_2} = \frac{p}{\delta}. \tag{4.39}$$

We shall apply this equation to the design of thin-walled vessels of most common shapes—spherical and cylindrical.

Spherical vessel. Let a thin-walled spherical vessel be subjected to a gauge internal pressure p (Fig. 53). Denote the mean diameter of the vessel by D. In a spherical vessel $\rho_1 = \rho_2 = D/2$.

On the basis of Eq. (4.39) we obtain

$$\frac{\sigma_1 + \sigma_2}{\frac{D}{2}} = \frac{p}{\delta}.$$

Since in a spherical vessel $\sigma_1 = \sigma_2 = \sigma$ because of symmetry, we have

$$\sigma = \frac{Dp}{4\delta}. \tag{4.40}$$

Consequently, the normal stress in the vessel is directly proportional to the pressure and diameter and inversely proportional to the wall thickness.

Cylindrical vessel. Let the mean diameter of a cylindrical vessel (Fig. 54) be D, its wall-thickness δ, its length l. Find the stresses σ_1 and σ_2 acting, respectively, at longitudinal and transverse sections, if the gauge internal pressure is p.

In a cylindrical vessel, the radius of curvature corresponding to a longitudinal section becomes infinite, $\rho_2 = \infty$, since the genera-

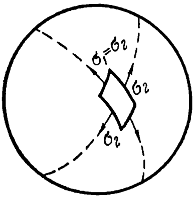


Fig. 53

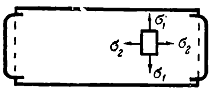


Fig. 54

tors (meridians) of the cylinder are straight lines, and the radius of curvature corresponding to a transverse section is equal to the radius of the cylinder

$$\rho_1 = \frac{D}{2}.$$

On the basis of Eq. (4.39) we obtain therefore

$$\frac{\sigma_1}{\frac{D}{2}} = \frac{p}{\delta}.$$

Consequently, the stress on a longitudinal section which tends to tear the vessel along the generator of the cylinder is

$$\sigma_1 = \frac{Dp}{2\delta}. \qquad (4.41)$$

As is seen from comparison with formula (4.40), this stress is twice as high as that in a spherical vessel of the same diameter.

Find now the stress on a transverse section of the cylindrical vessel. To do this, we cut the vessel by a plane perpendicular to the axis and remove one half. The pressure on the bottom of the vessel produces a force which tends to tear the vessel across a

transverse section. This force is equal to $p\frac{\pi D^2}{4}$. This force is balanced by the elastic force uniformly distributed over the transverse annular section of the vessel, i. e., by the force $\sigma_2\pi D\delta$. From the equilibrium condition we have

$$\sigma_2\pi D\delta = p\frac{\pi D^2}{4},$$

whence

$$\sigma_2 = \frac{Dp}{4\delta}. \tag{4.42}$$

From comparison of Eqs. (4.41) and (4.42) it is seen that the stress on a longitudinal section in a cylindrical vessel is twice as high as that on a transverse section. For this reason longitudinal riveted or welded joints of cylindrical vessels are made stronger than transverse joints.

Example 26. Using the energy strength theory, determine the wall thickness of a spherical vessel for storing compressed gas of pressure $p = 400$ kgf/cm^2 if $D = 40$ cm and the allowable stress $[\sigma] = 4{,}000$ kgf/cm^2.

Solution. The principal stresses are, according to formula (4.40),

$$\sigma_1 = \sigma_2 = \frac{pD}{4\delta}.$$

Substitute these stresses in the design formula of the fourth strength theory (4.37)

$$\sqrt{\left(\frac{pD}{4\delta}\right)^2 + \left(\frac{pD}{4\delta}\right)^2 - \frac{pD}{4\delta}\frac{pD}{4\delta}} < [\sigma],$$

whence

$$\delta \geqslant \frac{pD}{4[\sigma]} = \frac{400\times 40}{4\times 4{,}000} = 1 \text{ cm}.$$

In this case the design based on the fourth strength theory gives the same result as the design based on the first strength theory.

Example 27. Using the third and fourth strength theories, determine the wall thickness of a cylindrical vessel for gas of pressure $p = 150$ kgf/cm^2 if $D = 40$ cm and the allowable stress $[\sigma] = 4{,}000$ kgf/cm^2.

Solution. The principal stresses are, according to formulas (4.41) and (4.42),

$$\sigma_1 = \frac{pD}{2\delta}, \qquad \sigma_2 = \frac{pD}{4\delta}.$$

The compressive stress on the inside of the cylinder equal to the gas pressure in the vessel is neglected. By the third strength

theory, the wall thickness is

$$\delta \geqslant \frac{pD}{2[\sigma]} = \frac{150 \times 40}{2 \times 4{,}000} = 0.75 \text{ cm}.$$

To determine the wall thickness by the fourth strength theory, we substitute the values of σ_1 and σ_2 in formula (4.37)

$$\sqrt{\left(\frac{pD}{2\delta}\right)^2 + \left(\frac{pD}{4\delta}\right)^2 - \frac{pD}{2\delta}\frac{pD}{4\delta}} \leqslant [\sigma]$$

or

$$0.433\,\frac{pD}{\delta} \leqslant [\sigma],$$

whence

$$\delta \geqslant \frac{150 \times 40 \times 0.433}{4{,}000} = 0.65 \text{ cm}.$$

29. Check Questions

What formulas determine the normal and shearing stresses on inclined planes under simple tension?

At what sections do the maximum normal and the maximum shearing stresses occur in a rod under tension?

What is the sum of the normal stresses on two mutually perpendicular sections of a tension rod?

State the law of equal shearing stresses.

What are the principal stresses?

How are the principal stresses designated in the case of a three-dimensional state of stress?

What are a linear, plane and three-dimensional state of stress?

What formulas determine the principal stresses in the general case of plane stress?

Write the formulas for the maximum shearing stresses for a linear, plane and three-dimensional states of stress.

Express strains in terms of stresses for a three-dimensional state of stress.

Express the strain energy per unit volume in terms of stresses for a state of plane stress.

What are the strength theories used for?

What are the assumptions underlying the first, second, third and fourth strength theories?

State the strength theories suitable for ductile materials and those suitable for brittle materials.

Which of the strength theories are to be considered the more reliable? Substantiate this.

What is the general equation used for the design of thin-walled vessels?

Why are only the normal stresses σ_1 and σ_2 shown in isolating an element from the wall of a vessel (Fig. 52)?

Chapter V

Shear

30. Concept of Shear. Stresses in Shear. Hooke's Law in Shear

If a rod is acted on by two equal and opposite forces P, a very small distance apart and perpendicular to the axis of the rod, as in the case of cutting of metal bars or sheets with shears (Fig. 55*a*), shearing occurs provided the forces are sufficiently large. The left-hand part of the body is separated from the right-hand part

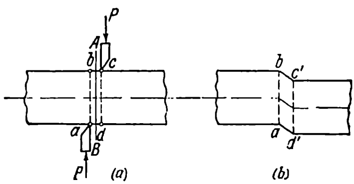


Fig. 55

through a section AB. The characteristic of shearing is the small distance between the forces P. The deformation preceding shearing consists in distortion of the right angles of an elementary parallelepiped. This deformation is called shear. Figure 55*b* shows the shear produced in the parallelepiped before shearing off; the rectangle *abcd* is changed into a parallelogram *abc'd'*. The distance *cc'* (Fig. 56) through which section *cd* is displaced relative to the adjacent section *ab* very close to it is called the *absolute shear*. The absolute shear depends on the distance between the adjacent sections *ab* and *cd*. The greater the distance (all other things being equal), the larger is the amount of the absolute shear.

The angle γ by which the right angles of the parallelepiped are changed is called the *angle of shear* or the *shearing strain*. In the elastic range this angle is very small. It will be recalled that strength of materials deals with small deformations produced in a material within the elastic limit.

The shearing strain can be determined from the relation

$$\frac{cc'}{ac} = \frac{a}{h} = \tan\gamma \cong \gamma. \tag{5.1}$$

Since the angle γ is small, its tangent may be taken to be equal to the angle itself.

A measure of shear is the shearing strain γ, i. e., the ratio of the absolute shear between two adjacent sections to the distance between these sections; it is expressed in radians.

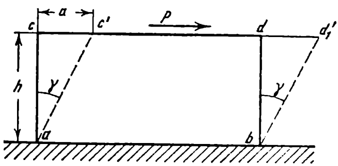


Fig. 56

If a section is passed in a rod between two shearing forces (Fig. 57*a*) and one part is removed, the action of the removed part on the remaining one should be replaced by internal forces. These forces act in the plane of the section (Fig. 57*b*). Consequently, shear deformation produces shearing stresses. If the internal forces are assumed to be uniformly distributed over the cross-sectional area, the magnitude of the shearing stresses is determined from the formula

$$\tau = \frac{P}{A}, \tag{5.2}$$

where A is the cross-sectional area of the rod. Experiments show that within the elastic limit the amount of shear is proportional to the shearing force P, the distance h over which the shear occurs and is inversely proportional to the cross-sectional area A.

Introducing a factor of proportionality $1/G$ depending on material properties, the law of elasticity for shear is expressed by the formula

$$a = \frac{Ph}{GA}. \tag{5.3}$$

Taking into account that $a/h = \gamma$ and $P/A = \tau$, we obtain an alternate expression of this law for shear

$$\tau = G\gamma. \tag{5.4}$$

Formula (5.4) is known as Hooke's law for shear. Comparing formula (5.2) with formula (2.2), formulas (5.3) and (2.4), and also (5.4) and (2.3) we see that *all fundamental formulas for shear are quite similar to those for tension and compression.*

The quantity G appearing in formulas (5.3) and (5.4) is called the *modulus of elasticity in shear* or the *modulus of elasticity of*

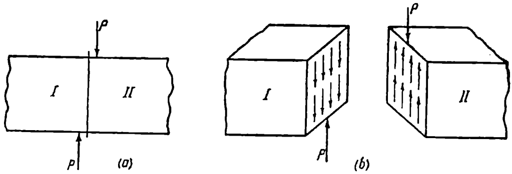


Fig. 57

the second kind. Since γ is a dimensionless quantity, it may readily be concluded from (5.4) that the dimension of G is the same as that of stress, i. e., kgf/cm². The quantities E and G for one and the same material are related by

$$G \cong 0.4E. \tag{5.5}$$

This relation is established experimentally. In Sec. 32 it will be obtained from theoretical considerations.

31. Pure Shear in a Rod Subjected to Tension and Compression in Two Mutually Perpendicular Directions

It is possible to produce a state of stress in a body such that the surfaces of elements suitably cut out from the body will be acted on by only shearing stresses. The state of stress in a right parallelepiped the four faces of which are acted on by only shearing stresses is called *pure shear.*

Consider a particular case when an element of a rod is subjected simultaneously to tensile and compressive forces in mutually perpendicular directions and the tensile and compressive stresses produced by these forces are numerically equal (Fig. 58*a*), i. e., $\sigma_1 = -\sigma_3 = \sigma$ and $\sigma_2 = 0$. Here we use the notation adopted in Sec. 23.

In Sec. 24 it was shown that the maximum shearing stresses occur on planes inclined at 45° and 135° to the principal planes. In this particular case the shearing stresses on these planes are, according to formula (4.10),

$$\tau_{\varphi\,\max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \frac{1}{2}(\sigma + \sigma) = \sigma.$$

There will be no normal stresses on these planes at all, as can readily be verified by substituting $\varphi=45°$ and $\varphi=135°$ in formula (4.8)

$$\sigma_\varphi=\sigma_1\cos^2 45°+\sigma_3\sin^2 45°=\frac{1}{2}\sigma_1+\frac{1}{2}\sigma_3=\frac{1}{2}\sigma-\frac{1}{2}\sigma=0,$$

$$\sigma_\varphi=\sigma_1\cos^2 135°+\sigma_3\sin^2 135°=\frac{1}{2}\sigma_1+\frac{1}{2}\sigma_3=\frac{1}{2}\sigma-\frac{1}{2}\sigma=0.$$

Consequently, in the case under consideration an element *abcd* inside the rod, the lateral faces of which make angles of 45° with

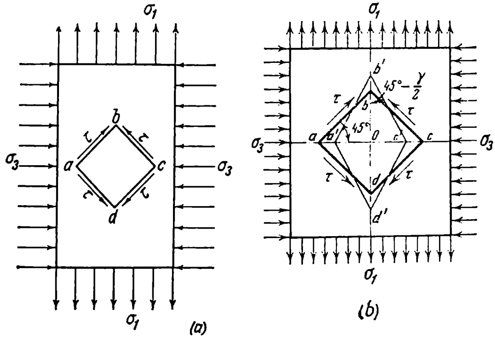


Fig. 58

the directions of the principal stresses, is subjected to only shearing stresses on its four faces. In other words, the element is in a state of stress called pure shear.

32. Relation Between Moduli of Elasticity E and G

It was shown above that the extension or compression of a rod is accompanied by shear on inclined planes in the rod. Consequently, the tensile or compressive deformation is closely related to the shearing deformation. On the basis of this relationship it is possible to determine theoretically the relation between the moduli of elasticity E and G.

Take a right parallelepiped stretched in one direction and compressed in the other, as shown in Fig. 58*b*.

If $\sigma_1=\sigma$, $\sigma_2=0$ and $\sigma_3=-\sigma$, an element inside the body is in a state of pure shear. The lengths *Ob* and *Oc* equal before deformation are changed after deformation: the length *Ob* increases and

becomes Ob' while the length Oc decreases and becomes Oc'

$$Ob' = Ob(1+\varepsilon_1), \quad Oc' = Oc(1+\varepsilon_3).$$

The strains ε_1 and ε_3 are, by formulas (4.16),

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \mu\sigma_3) = \frac{1}{E}(\sigma + \mu\sigma) = \frac{\sigma}{E}(1+\mu).$$

$$\varepsilon_3 = \frac{1}{E}(\sigma_3 - \mu\sigma_1) = \frac{1}{E}(-\sigma - \mu\sigma) = -\frac{\sigma}{E}(1+\mu).$$

Consequently,

$$Ob' = Ob\left[1 + (1+\mu)\frac{\sigma}{E}\right],$$

$$Oc' = Oc\left[1 - (1+\mu)\frac{\sigma}{E}\right].$$

The right angle between the faces ab and bc of the element decreases. Denote the decrease in this angle, i. e., the shearing strain, by γ; then

$$\tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) = \frac{Oc'}{Ob'} = \frac{1-(1+\mu)\frac{\sigma}{E}}{1+(1+\mu)\frac{\sigma}{E}}. \qquad \text{(a)}$$

Since $\gamma/2$ is a small angle, we may put $\tan(\gamma/2) = \gamma/2$. Therefore

$$\tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) = \frac{\tan\frac{\pi}{4} - \tan\frac{\gamma}{2}}{1 + \tan\frac{\pi}{4}\tan\frac{\gamma}{2}} = \frac{1-\frac{\gamma}{2}}{1+\frac{\gamma}{2}}. \qquad \text{(b)}$$

From the formulas (a) and (b) we obtain

$$\frac{1-\frac{\gamma}{2}}{1+\frac{\gamma}{2}} = \frac{1-(1+\mu)\frac{\sigma}{E}}{1+(1+\mu)\frac{\sigma}{E}}$$

or

$$\gamma = 2(1+\mu)\frac{\sigma}{E}. \qquad \text{(c)}$$

In Sec. 31 it was shown that the shearing stresses τ on the faces of an elementary cube which is in a state of pure shear are equal to the stress σ, i. e., $\tau = \sigma$. On the other hand, $\tau = G\gamma$. Consequently, in this case

$$\sigma = G\gamma.$$

Substituting this value of σ in the formula (c), we obtain

$$G = \frac{E}{2(1+\mu)}. \qquad \text{(5.6)}$$

This formula expresses the relation between the modulus of elasticity in shear and the modulus of elasticity in tension or compression.

The numerical relation between G and E varies with the value of Poisson's ratio μ for a given material.

If $\mu=\frac{1}{4}$, $G=\frac{2}{5}E$.

If $\mu=\frac{1}{3}$, $G=\frac{3}{8}E$.

Table 7 gives average values of the modulus of elasticity G for some materials.

Table 7. Modulus G

Material	G, kgf/cm²	Material	G, kgf/cm²
Steel Cast iron Copper	8.1×10^5 $(3 \text{ to } 4.5)\times10^5$ 4×10^5 to 4.9×10^5	Aluminium Wood	2.6×10^5 0.055×10^5

33. Allowable Stress in Shear

The problem of choosing the allowable stress in shear is more complicated than in the case of tension and compression. The choice of the allowable stress is based on the yield strength or the ultimate strength of a material. However, a direct determination of these characteristics of a material in shear is complicated by the fact that in practice it is difficult to reproduce pure shear without bending effect or any other side phenomena affecting test results. Therefore, the allowable stress in shear is established from theoretical considerations supported by experience.

In Sec. 25 we considered the general case of plane stress shown in Fig. 48.

The principal stresses for this general case are determined by Eqs. (4.14)

$$\left.\begin{aligned}\sigma_{max}&=\frac{\sigma_x+\sigma_y}{2}+\frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2},\\ \sigma_{min}&=\frac{\sigma_x+\sigma_y}{2}-\frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2}.\end{aligned}\right\} \qquad (4.14)$$

Pure shear is a particular case of the general case of plane stress when $\sigma_x=\sigma_y=0$, i. e., when the faces of an element are acted on by only shearing stresses. Therefore, the principal stresses for the case of pure shear, as determined from formulas (4.14), are

$$\sigma_{max}=\tau, \qquad \sigma_{min}=-\tau.$$

In a state of plane stress one of the three principal stresses is zero. Since σ_{min} is negative, it is the smallest of the three principal stresses and must be denoted by σ_3 (see Sec. 23).

Thus, in pure shear

$$\sigma_1 = \tau, \quad \sigma_2 = 0, \quad \sigma_3 = -\tau.$$

According to the first strength theory (4.22) we obtain the condition

$$[\sigma] \geqslant \sigma_1 = \tau,$$

i. e., the shearing stress must not exceed the allowable stress in tension, i. e.,

$$[\tau] \leqslant [\sigma].$$

According to the second strength theory (4.26) we have

$$[\sigma] \geqslant \sigma_1 - \mu\sigma_3 = (\tau + \mu\tau).$$

If for steel $\mu = 0.3$, the allowable shearing stress must be

$$[\tau] \cong 0.77\,[\sigma].$$

Likewise, according to the third strength theory (4.32) we obtain

$$[\sigma] \geqslant \sigma_1 - \sigma_3 = \tau + \tau = 2\tau.$$

Consequently,

$$[\tau] = 0.5\,[\sigma].$$

Finally, according to the fourth (energy) strength theory (4.38) we have

$$[\sigma] \geqslant \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)} = \sqrt{\tau^2 + \tau^2 + \tau^2} = \tau\sqrt{3}.$$

Consequently,

$$[\tau] = 0.57\,[\sigma].$$

In practice it is customary to assume for brittle materials

$$[\tau] = (0.7 \text{ to } 1.0)\,[\sigma], \tag{5.7}$$

for ductile materials

$$[\tau] = (0.5 \text{ to } 0.6)\,[\sigma]. \tag{5.8}$$

Fibrous materials, such as wood, resist shear parallel to grain in a way different from isotropic materials: they undergo shearing between fibres. The choice of the allowable stress for such materials is made on the basis of available experiments.

Thus, for pine of medium quality the allowable stress in shear parallel to grain is $[\tau] \cong 10$ kgf/cm^2 and the allowable stress in tension parallel to grain is $[\sigma] = 100$ kgf/cm^2. Hence, $[\tau]$ for pine is only $0.1\,[\sigma]$.

If the allowable stress in shear is known, it is easy to write the strength condition (design equation) for shear

$$\tau = \frac{P}{A} \leqslant [\tau]. \tag{5.9}$$

This equation is of the same form as the design equation for tension (compression).

34. Crushing

Shear deformation is often accompanied by crushing. The characteristic of crushing is the action of a compressive force over a relatively small area. If, for instance, two wooden rods are bolted together (Fig. 59*a*), the wood surface under the nut and under the

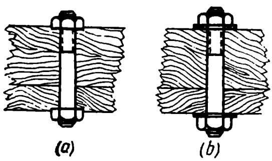


Fig. 59

bolt head is compressed and pressed in when the nut is screwed. The stress due to the localized compression decreases rapidly with the distance from this region. To reduce the bearing stress, the bearing surface is increased by inserting metal washers under the nut and the bolt head (Fig. 59*b*), which have a larger area of contact with the wood.

If the contact surfaces are well finished, the bodies make contact at each point of the joint; in this case central compressive forces may be expected to produce a uniform distribution of bearing stres es over the entire joint.

Denoting the compressive force by P, the bearing area by A and the bearing stress by σ_b, we obtain

$$\sigma_b = \frac{P}{A}. \tag{5.10}$$

A check for crushing is carried out for the softer material if the bodies in contact are made of dissimilar materials.

For uniformly distributed bearing stresses, the allowable stress $[\sigma_b]$ for steel is taken as

$$[\sigma_b] = (2 \text{ to } 2.5)\,[\sigma],$$

where $[\sigma]$ is the allowable stress in compression. For wood, $[\sigma_b]$ depends on the direction of the force relative to the direction of fibres and ranges from 0.25 to 0.8 of $[\sigma]$.

Relating the allowable bearing stress to the actual one, we write the strength equation as

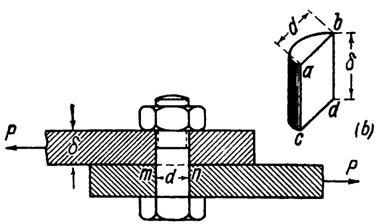
$$\sigma_b = \frac{P}{A} \leqslant [\sigma_b]. \tag{5.11}$$

35. Examples of Design for Shear and Crushing

In real conditions, shear is usually accompanied by crushing and bending with the resulting normal stresses. The shearing stresses on sections where shear occurs are often non-uniformly distributed as are the bearing stresses. For simplicity, however, the shearing and bearing stresses are assumed to be uniformly distributed over sections. Thus, such a simplifying assumption is made in the design of rivets, keys, slits, cuts, etc.

The stresses determined under simplifying assumptions differ from the actual values—they are conventional. The choice of these conventional stresses is based on observations of the functioning of completed structures.

Consider several examples of design for shear and crushing.



(a)

Fig. 60

Example 28. Design the bolted joint shown in Fig. 60*a* if the force $P = 800$ kgf, the thickness of the parts being joined together is $\delta = 8$ mm, the allowable shearing stress $[\tau] = 600$ kgf/cm² and the allowable bearing stress $[\sigma_b] = 2{,}000$ kgf/cm².

Solution. The force P tends to shear off the bolt along a section *mn*. The required bolt diameter is determined from the design

equation (5.9)

$$\frac{P}{A} = \frac{P}{\frac{\pi}{4} d^2} \leqslant [\tau],$$

whence

$$d \geqslant \sqrt{\frac{4P}{\pi [\tau]}} = \sqrt{\frac{4 \times 800}{3.14 \times 600}} = 1.3 \text{ cm}.$$

Check the hole walls in the bolted parts for crushing. The bearing surface due to the pressure exerted by the bolt on the hole wall in one part represents the lateral surface of a half-cylinder of height δ and diameter d (Fig. 60*b*). The pressure distribution over the surface of this half-cylinder is not known; it depends on the clearance between the bolt and the parts being joined together, on the inaccuracy in the shape of the hole and bolt, as well as on the kind of material. For simplicity, it is customary to assume that the pressure is uniform and equal in magnitude to the force divided not by the surface of the half-cylinder but by its projection *abcd* on the diametral plane, i. e., by the area δd.

Thus, the bearing stress in the hole walls is

$$\sigma_b = \frac{P}{\delta d} = \frac{800}{0.8 \times 1.3} = 768 \text{ kgf/cm}^2.$$

The resulting stress is smaller than the allowable value, i. e., $\sigma_b < [\sigma_b]$. If the bearing stress is greater than the allowable value, the bolt crushes the surfaces of the parts and in consequence the

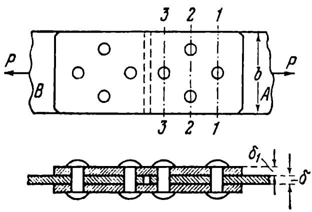


Fig. 61

operating conditions of the bolt are impaired. Therefore, the hole walls must always be checked for crushing.

Example 29. Two steel strips, A and B, of thickness $\delta = 10$ mm and width $b = 150$ mm are riveted together with two cover plates as shown in Fig. 61. The diameter of rivets is $d = 16$ mm, the thickness of cover plates is $\delta_1 = 6$ mm and their width is equal to

the width of the riveted strips, i. e., 150 mm. Determine the stresses in the rivets and strips if the tensile load $P=8$ tons.

Solution. (1) Stress in rivets. There are four rivets ($i=4$) on either side of the butt joint; each of the rivets is sheared off by the force P/i on two sections (double-shear rivets); consequently, the shearing stress in the rivets is

$$\tau=\frac{P}{i2\frac{\pi}{4}d^2}=\frac{8{,}000}{4\times 2\times 0.785\times 1.6^2}\cong 500 \text{ kgf/cm}^2.$$

(2) Stress in riveted strips. At section 1-1 the entire force P is transmitted by the sheet. This section is weakened by one rivet; therefore, the cross-sectional area is

$$A_{1-1}=\delta(b-d)=1(15-1.6)=13.4 \text{ cm}^2.$$

The tensile stress on this section is

$$\sigma_{1-1}=\frac{P}{A_{1-1}}=\frac{8{,}000}{13.4}=597 \text{ kgf/cm}^2.$$

At section 2-2 the transmitted force is only $\frac{3}{4}P$ since the force $\frac{1}{4}P$ is carried by the first rivet and transmitted to strip B through the cover plates and the rivets located to the left of the joint. Section 2-2 is weakened by two rivets; consequently, its design area is

$$A_{2-2}=\delta(b-2d)=1(15-2\times 1.6)=11.8 \text{ cm}^2.$$

The tensile stress on this section is

$$\sigma_{2-2}=\frac{\frac{3}{4}P}{A_{2-2}}=\frac{\frac{3}{4}\times 8{,}000}{11.8}=508 \text{ kgf/cm}^2,$$

i. e., 15 per cent less than on section 1-1.

At section 3-3 the force transmitted to strip B is $P-\frac{3}{4}P=\frac{1}{4}P$.

The design cross-sectional area of the strip is the same as at section 1-1, i. e.,

$$A_{3-3}=A_{1-1}=13.4 \text{ cm}^2.$$

The tensile stress on this section is

$$\sigma_{3\text{-}3}=\frac{\frac{1}{4}P}{A_{3\text{-}3}}=\frac{\frac{1}{4}\times 8{,}000}{13.4}=150 \text{ kgf/cm}^2,$$

i.e., 75 per cent less than at section 1-1.

Determine now the bearing stress in the strip due to the pressure exerted by the rivet.

The bearing area in the strip per one rivet is

$$A_b = \delta d = 1 \times 1.6 = 1.6\ \text{cm}^2.$$

The pressure exerted by one rivet is $\frac{1}{4}P$; consequently, the bearing stress is

$$\sigma_b = \frac{\frac{1}{4}P}{A_b} = \frac{\frac{1}{4}\times 8{,}000}{1.6} = 1{,}250\ \text{kgf/cm}^2.$$

Example 30. Determine the force P (Fig. 62) which must be applied to a punch to make a hole of diameter $d = 12$ mm in a steel sheet of thickness $\delta = 10$ mm if the ultimate shearing strength of the sheet material is $\tau_{u.s} = 4{,}000$ kgf/cm².

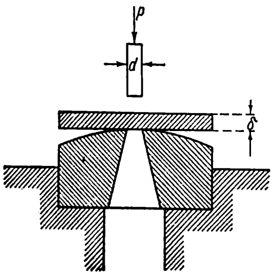


Fig. 62

Solution. The area sheared off by the punch in perforating the sheet is

$$A = \pi d\delta = 3.14 \times 1.2 \times 1 = 3.78\ \text{cm}^2.$$

The force P required to make a hole is

$$P = \tau_{u.s}A = 4{,}000 \times 3.78 = 15{,}120\ \text{kgf}.$$

Example 31. Design the connection of a spar to a tie beam; the material is pine (Fig. 63). The angle between the axes of the spar and tie beam is $\alpha = 30°$. The force acting along the spar is $P = 4{,}000$ kgf; the allowable stress for pine in shear is $[\tau] = 12$ kgf/cm², in crushing at an angle $[\sigma_b] = 60$ kgf/cm²; the cross-sectional dimensions of the spar are $h = b = 15$ cm.

Solution. The force tending to shear off the end of the tie beam along a length x and crush the undercut on an area *ncdm* is

$$P_1 = P\cos 30° = 4{,}000 \times 0.866 = 3{,}460\ \text{kgf}.$$

The required bearing area of the undercut is

$$A_b \geqslant \frac{P_1}{[\sigma_b]} = \frac{3{,}460}{60} = 58\ \text{cm}^2.$$

The depth of the undercut is

$$y = \frac{A_b}{b} = \frac{58}{15} = 3.86\ \text{cm}; \quad \text{take 4 cm.}$$

The length x of the tie beam extending beyond the undercut is determined from the condition that the shearing area $A_{sh}=xb$ be sufficient to ensure the strength of the connection.

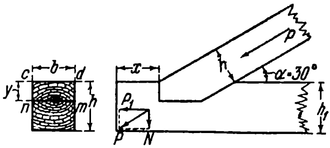


Fig. 63

The required shearing area is

$$A_{sh} \geqslant \frac{P_1}{[\tau]} = \frac{3,460}{12} = 290 \text{ cm}^2.$$

From the condition that $xb=290$ we determine the length

$$x = \frac{290}{b} = \frac{290}{15} = 19.4 \text{ cm};$$

rounding off, we take $x=20$ cm.

36. Design of Welded Joints

In the last few decades electric welding has found very wide application. Welded joints are not so labour-consuming as riveted joints, they do not weaken sections of the elements being connected, they simplify the construction and are just as reliable as riveted joints. That is why riveted joints are progressively superseded by welded joints in engineering practice.

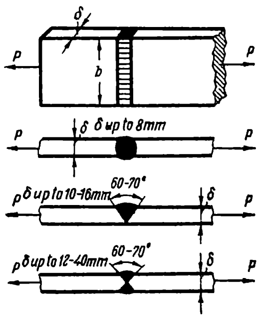


Fig. 64

The basic and simplest type of welded joint is a butt joint (Fig. 64) in which the clearance between the two elements is filled with metal. The clearance between the elements can be shaped as shown in Fig. 64, depending on the thickness of the welded elements.

The joint is checked for tension or compression using the equation

$$\sigma = \frac{P}{l\delta} \leqslant [\sigma_e], \tag{5.12}$$

where l is the design length of the weld, δ is the thickness of the welded elements and $[\sigma_e]$ is the allowable stress in electric welding.

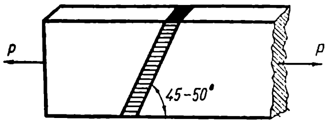


Fig. 65

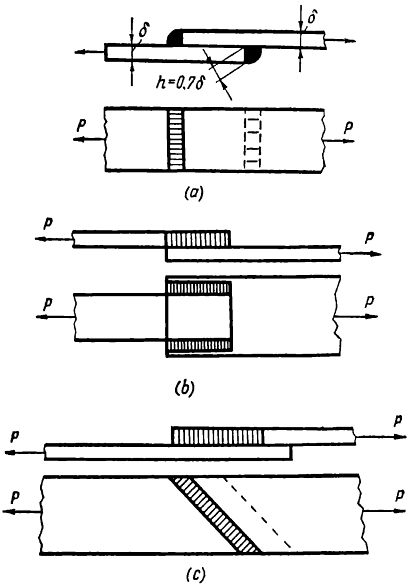


Fig. 66

The design length of a weld is always taken 10 mm less than the actual length because of welding defects at the ends. The depth of a weld is somewhat greater than the thickness of the

welded elements but, to be on the "safe" side in design, it is taken equal to the thickness δ.

As experiments show, a butt joint with an oblique weld (Fig. 65) at 45° is as strong as a solid section.

Another type of joint is a lap joint made by means of fillet welds. Fillet welds are called transverse fillets when they are perpendicular to the direction of the acting force (Fig. 66*a*), side fillets when they are parallel to the direction of the force (Fig. 66*b*) and oblique fillets when they are at an angle to the direction of the force (Fig. 66*c*).

Fillet welds are designed for shear on a section of area $A = lh$, where

$$l = b - 10 \text{ mm}, \quad h = \delta \cos 45^\circ \cong 0.7\delta.$$

The strength condition is

$$\tau = \frac{P}{0.7l\delta} \leqslant [\tau_e], \tag{5.13}$$

where $[\tau_e]$ is the allowable shearing stress in electric welding. If the sheets are connected as shown in Fig. 66*a*, the force is carried by two welds and the strength condition becomes

$$\tau = \frac{P}{1.4l\delta} \leqslant [\tau_e]. \tag{5.14}$$

The choice of allowable stresses for welds is governed by the material of structural elements to be connected and the method of welding; they are given in Table 8.

Table 8. Allowable Stresses for Electric Welds

Type of Stress	Symbol	Manual Welding		Automatic Welding (kgf/cm²)
		Electrodes with Thin Coat (kgf/cm²)	Electrodes with Thick Coat (kgf/cm²)	
Tension	$[\sigma_e]$	1,000	1,300	1,300
Compression	$[\sigma_e]$	1,100	1,450	1,450
Shear	$[\tau_e]$	800	1,100	1,100

Example 32. Determine the allowable force for the butt weld shown in Fig. 67. The allowable stress for the base metal is $[\sigma] = 1{,}400$ kgf/cm², for the weld metal $[\sigma_e] = 1{,}000$ kgf/cm².

Solution. The allowable force carried by the weld is determined by formula (5.12)

$$P \leqslant [\sigma]\, lt.$$

As stated above, the design length of the weld is taken 1 cm less than the actual length, taking into account poor penetration at the ends,

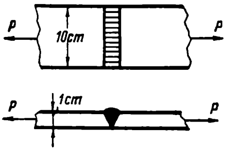


Fig. 67

$$l = 10 - 1 = 9 \text{ cm}.$$

The design thickness of the butt weld is taken equal to the thickness of the connected strips, $t = 1$ cm. The allowable force is

$$P \leqslant 1{,}000 \times 9 \times 1 = 9{,}000 \text{ kgf}.$$

The allowable force for the base metal is

$$P \leqslant [\sigma]\, bt = 1{,}400 \times 10 \times 1 = 14{,}000 \text{ kgf}.$$

Consequently, the base metal is underloaded in the case of the weld considered.

Example 33. Determine the required dimensions of side fillet welds (Fig. 68) connecting strips of the same cross-sectional area as in the preceding example. The tensile force is $P = 14{,}000$ kgf

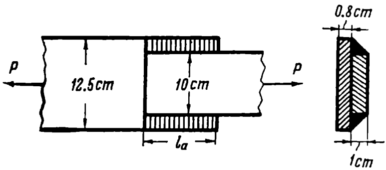


Fig. 68

and the allowable stress for the weld metal in shear is $[\tau_e] = 800$ kgf/cm².

Solution. Determine the actual length of the weld l_a. By formula (5.14), the design length of the weld is

$$l \geqslant \frac{P}{1.4t\,[\tau_e]} = \frac{14{,}000}{1.4 \times 1 \times 800} = 12.5 \text{ cm}.$$

The actual length of the weld is

$$l_a = 12.5 + 1 = 13.5 \text{ cm}.$$

37. Check Questions

Under what conditions does shearing occur?
What is the shear deformation?

What are the absolute shear and the shearing strain? What are their dimensions?

What formula determines stresses in shear?

What state of stress is called pure shear?

State Hooke's law in shear.

What is the relation between the moduli of elasticity of the first and second kind?

What is the relation between the allowable stresses in tension and in shear?

What is crushing?

What types of welded joint do you know?

What is the difference between a single-shear and a double-shear rivet?

Chapter VI

Torsion

38. Construction of Twisting Moment Diagrams. Relation Between Torque, Power and Number of Revolutions

Many machine and structural parts, such as shafts, springs, etc., are subjected to torsion. By way of example we consider the torsion of a shaft. Imagine a shaft (Fig. 69) supported by bearings with two pulleys, A and B, mounted on it. Pulley A is connected

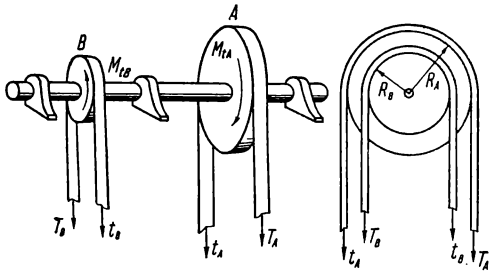


Fig. 69

to the motor by a belt drive, and pulley B is connected to the machine by a belt drive. The rotation transmitted to the shaft by pulley A is produced by unequal tensions in the part of the belt which pulls and in the part which resists. The tension in the tight part of the belt T_A is larger than the tension in the loose part of the belt t_A.

The tensions T_A and t_A in the belt passing over pulley A produce, in addition to the pressure on supports, a couple which can easily be determined by setting up the moment equation with respect to the centre of the pulley

$$M_{tA}=T_AR_A-t_AR_A=(T_A-t_A)\,R_A.$$

The machine driven by pulley B resists the rotation, therefore the tensions developed in the parts of the belt passing over this

pulley form the couple

$$M_{tB}=T_BR_B-t_BR_B=(T_B-t_B)\,R_B.$$

If friction in the bearings is neglected, it follows from the condition of equilibrium of the shaft in the steady-state uniform rotation that the moment M_{tA} provided by the motor must be equal to the moment M_{tB} due to the useful resistance, i.e.,

$$(T_A-t_A)\,R_A=(T_B-t_B)\,R_B=M_t.$$

The portion of the shaft between the pulleys, i.e., the portion between the planes where the couples are transmitted to the shaft,

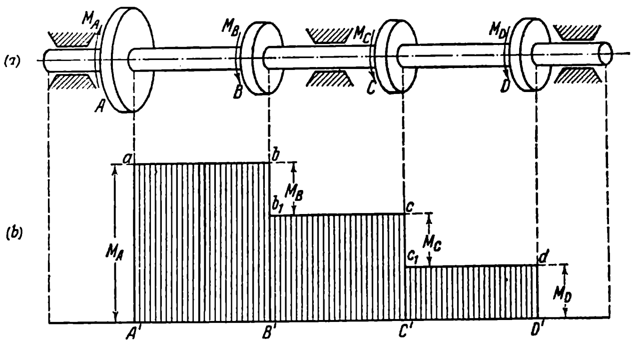


Fig. 70

is twisted. The shaft remains in this twisted condition all the time during operation. After the motor is stopped, the twisting couples cease to act, the deformation vanishes and the shaft comes untwisted.

To determine the stresses and strains in the shaft, it is necessary to know the magnitudes of twisting moments acting in its separate portions. A diagram showing the magnitudes of twisting moments along the length of the shaft is called a *twisting moment diagram*.

We proceed to the construction of twisting moment diagrams.

Suppose that a transmission shaft (Fig. 70*a*) receives a twisting moment M_A from the motor by means of a belt drive and pulley A. This twisting moment is used in overcoming the resistance to the rotation of parts of running machines via transmitting pul-

leys B, C and D connected to the machines by belts. Write the equilibrium condition

$$M_A = M_B + M_C + M_D.$$

The twisting moments acting in the portions of the shaft between the pulleys have different magnitudes. To plot a twisting moment diagram, we draw a line $A'D'$ (Fig. 70*b*) parallel to the axis of the shaft. Points A' and D' correspond to the middle planes of the extreme pulleys. From point A' we erect a perpendicular whose length $A'a$ represents, to a chosen scale, the twisting moment M_A received by the shaft from the motor. This moment remains unchanged up to the middle plane of pulley B. Therefore, from point a we draw a line ab parallel to $A'D'$. Part of the twisting moment M_A, namely, a twisting moment M_B, is transmitted to the machine through pulley B. Consequently, the twisting moment remaining on the shaft beyond the middle plane of pulley B is $M_A - M_B$. To this twisting moment corresponds an ordinate of the diagram $B'b_1$.

The twisting moment $M_A - M_B$ remains unchanged up to the middle plane of pulley C. Therefore, from point b_1 we draw a line b_1c parallel to $A'D'$. The twisting moment M_C transmitted to the machine through pulley C is represented to scale by a segment cc_1. Consequently, the moment remaining on the shaft beyond pulley C is $M_A - M_B - M_C$. This moment represented in the diagram by ordinate $C'c_1$ remains unchanged up to the middle plane of the last pulley D. Therefore, from point c_1 we draw a line c_1d parallel to $A'D'$. Pulley D transmits to the machine a twisting moment M_D equal to the twisting moment remaining on the shaft, $M_A - M_B - M_C$. Beyond pulley D, just as ahead of pulley A, the twisting moment is zero.

The twisting moment diagram represented by the stepped line $A'abb_1cc_1dD'$ shows that the most heavily loaded portion of the shaft is the portion between pulleys A and B which transmits the largest twisting moment. If a shaft of circular section has the same cross-sectional area throughout its length, the diameter is determined by the maximum twisting moment. In this case the portions of the shaft with smaller twisting moments have excessive strength. Therefore, theoretically it is more advantageous to make a shaft of variable diameter along its length. However, the advantage is not always realized in practice because of an increase in cost of fabrication of the shaft and the presence of stress concentration at transitions from one thickness to another. Saving in material can be effected by rational arrangement of pulleys on the shaft, namely, it is more advantageous to place a pulley receiving the twisting moment from the motor in the middle part of the shaft so that the sums of moments distributed by the shaft

on either side of this pulley are the same. This will be illustrated by the following example.

Let a twisting moment M_A be transmitted from the motor to a shaft via a pulley A (Fig. 71*a*) located between pulleys B and C on the right and D on the left which transmit, respectively, moments M_B, M_C and M_D from the shaft to operating machines. Neglecting friction in the bearings, we can write

$$M_A = M_B + M_C + M_D$$

To construct twisting moment diagrams correctly, it should be remembered that *the moment acting at any section of a shaft is*

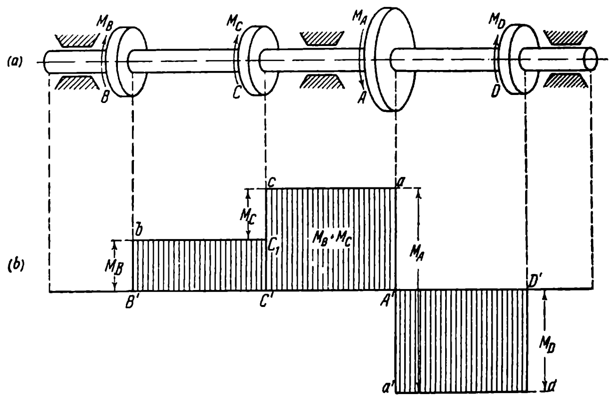


Fig. 71

equal to the sum of twisting moments lying to one side of this section. Consider the twisting moments distributed by the shaft as shown in Fig. 71.

In this case, the torque in the portion of the shaft from the left support to the middle plane of pulley B is zero. The twisting moment at any section of portion BC is equal to the moment transmitted by pulley B, i. e., to the moment M_B. This moment is represented to scale in Fig. 71*b* by a segment $B'b$ laid off upward from the axis of abscissas. The twisting moment at any section of portion CA of the shaft is equal to the sum of twisting moments to the left of the section, i. e., $M_B + M_C$. This moment is represented in the diagram by segment $C'c$. The twisting moment M_A transmitted to

the shaft at the middle plane of pulley A has a sense opposite to that of the moments M_B and M_C. Therefore, the twisting moment acting in portion AD of the shaft is equal to $M_B+M_C-M_A$. Since $M_A>M_B+M_C$, the moment diagram crosses the axis of abscissas at section A of the shaft. The moment M_D released from the shaft at the middle plane of pulley D is equal in magnitude to the moment $M_B+M_C-M_A$ but has an opposite sense. Therefore, from point d of the moment diagram we draw a line dD' upward. The twisting moment to the right of section D of the shaft is zero. The stepped line $B'b\,C_1caa'dD'$ represents the twisting moment diagram.

If a pulley receiving a twisting moment were placed to one side of transmitting pulleys B, C and D, the twisting moment diagram would be of the form shown in Fig. 70*b*. From comparison of the diagrams of Figs. 70*b* and 71*b* it is easily seen that when the pulley receiving the torque from the motor is placed between the distributing pulleys the maximum twisting moment is considerably smaller than when the pulley receiving the torque from the motor is placed to one side of the transmitting pulleys. A reduction in the maximum twisting moment on the shaft results, of course, in a reduction of the shaft diameter and hence in saving of material.

In the design of shafts, it is customary to assign the power H (horsepower) transmitted by the shaft and the number of revolutions n of the shaft rather than the torque. Let us derive a formula for determining the torque from the assigned power H (hp) and the number of revolutions n (rpm). From mechanics it is known that the power produced by the torque is equal to the torque multiplied by the angular velocity, i. e.,

$$N=M_t\omega=M_t\frac{\pi n}{30}\ \text{kgf-m/sec}.$$

On the other hand,

$$N=75H\ \text{kgf-m/sec};$$

consequently,

$$M_t\frac{\pi n}{30}=75H,$$

whence

$$M_t=\frac{30\times 75}{\pi}\frac{H}{n}=716.2\frac{H}{n}\ \text{kgf-m}$$

or

$$M_t=71{,}620\frac{H}{n}\ \text{kgf-cm}. \tag{6.1}$$

It will be recalled that in the above formula H stands for the horsepower and n the number of revolutions per minute.

39. Determination of Stresses and Strains in a Circular Bar Subjected to Torsion

Before proceeding to the derivation of equations for determining stresses and strains under torsional loading, we shall discuss some experimental results.

Assume that a circular cylinder the lower end of which is attached to a fixed plane N (Fig. 72*a*) is subjected at its free upper end to a couple of moment M_t applied in a plane perpendicular to the axis of the cylinder. Under the action of this moment, the cylinder undergoes *twisting deformation*. The axis O-O of the cy-

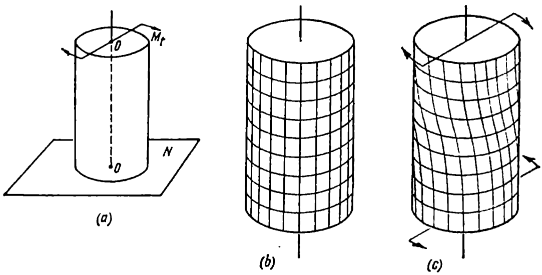


Fig. 72

linder remains straight during twisting. This axis is called the *axis of torsion*. If, before twisting, a network of equidistant circles and generators (Fig. 72*b*) is marked on the lateral surface of a cylinder, the following occurs (Fig. 72*c*) under small deformations.

(1) The squares formed by the network are changed into identical rhombi.

(2) The circular sections of the cylinder remain circular with the same diameter.

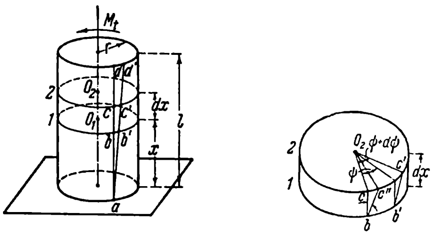
(3) The distances between circles are unaltered and hence the total length of the cylinder remains unchanged.

(4) The generators of the cylinder turn into helices of large lead.

It is of course impossible to assess with absolute assurance the changes occurring at the interior points of the cylinder in torsion from these outward appearances. But the fact that the circles marked on the cylinder and the ends of the cylinder remain plane after deformation and the generators turn into helices provides justification for assuming that each cross section remains plane and rotates with respect to adjacent cross sections. The rotation of cross sections about the axis of the cylinder through a certain angle

occurs as if the cross sections were absolutely rigid bodies. As experiments show, the angles of rotation of cross sections about their centres are directly proportional to their distances from the fixed end. The angle of rotation of the end section is called the *total angle of twist*. The theoretical conclusions based on the assumption that the cross sections of a circular cylinder remain plane during twisting are fully supported by experimental investigations.

The theory of torsion of circular bars is based on the following three assumptions: (1) plane cross sections of a rod remain plane



(a) (b)

Fig. 73

during deformation, (2) the radii of cross sections remain straight, (3) the distances between cross sections remain unchanged.

We proceed to the derivation of basic equations of torsion of circular bars. Isolate a disk of radius r from a twisted bar (Fig. 73*a*) at a distance x from the fixed end by two adjacent cross sections, *1* and *2*, which are dx apart. Points b, c and d, lying on the same generator before deformation, occupy new positions b', c', d' on a helix after deformation.

If section 1 a distance x from the lower end rotates through an angle ψ with respect to the lower end, section 2 which is at a distance $x+dx$ rotates through an angle $\psi+d\psi$ with respect to the fixed end (Fig. 73*b*).

Draw from point b a straight line bc'' parallel to $b'c'$ and join the centre of the second section to point c''. The angle cO_2c'' equal to $d\psi$ is then the angle of rotation of section 2 with respect to section 1. The lateral sides of the element $bc''c'b'$ were vertical before section 2 was rotated with respect to section 1. After rotation the sides become inclined and assume positions bc'' and $b'c'$. Consequently, the element undergoes an absolute shear equal to the arc length

$$cc'' = r d\psi.$$

The shearing strain of the element is

$$\gamma = \frac{r\,d\psi}{dx}.$$

The ratio $d\psi/dx$ represents the angle of twist per unit length of the rod. Denote it by θ. Then

$$\gamma = r\theta. \tag{6.2}$$

From this formula it is evident that the shearing strain is proportional to the radius of a twisted cylindrical body.

Since we have assumed at the beginning of this paragraph that the radii of cross sections do not distort but remain straight, it may be said that, for an element similar to the isolated one but lying inside the cylinder at a distance ρ from the centre, the shearing strain is

$$\gamma_\rho = \rho\theta. \tag{a}$$

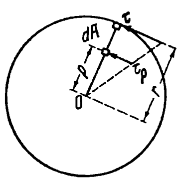


Fig. 74

On the basis of Hooke's law for shear

$$\tau = G\gamma \tag{5.4}$$

it is possible to determine the stress for any element of a body from its shearing strain. Thus, for elements situated at the surface of a bar the stress is, according to Eqs. (6.2) and (5.4),

$$\tau = G\theta r.$$

For an element a distance ρ from the axis (Fig. 74), the stress is

$$\tau_\rho = G\theta\rho. \tag{b}$$

The elementary tangential force on the area dA of the cross section is

$$\tau_\rho dA = G\theta\rho dA.$$

The direction of these internal elementary forces is perpendicular to the corresponding radii since it is in this direction that shearing takes place.

The moment of the elementary force about the axis of the rod is

$$dM = G\theta\rho^2 dA. \tag{c}$$

The sum of these elementary moments extended over the entire cross section A must be equal to the external torque at equilibrium after deformation. This sum of moments is found by integ-

rating the expression (c)

$$M_t=\int_A dM=\int_A G\theta\rho^2 dA.$$

Putting the constants before the integral sign, we obtain

$$M_t=G\theta\int_A \rho^2 dA. \tag{d}$$

The expression $\int_A \rho^2 dA$, i. e., the sum of products of elementary areas (dA) and the squares of their distances (ρ^2) to some pole lying in the plane of the figure, which is extended over the entire area of the figure, is called the *polar moment of inertia* of the figure and denoted by I_p.

The polar moment of inertia is a geometric quantity with the dimension cm^4. The polar moment of inertia is always a positive quantity. In our derivation the pole for I_p is the centre of the section, i. e., the centre of the circle.

Introducing the notation for the polar moment of inertia in the expression (d), we obtain

$$M_t=G\theta I_p,$$

whence the angle of twist per unit length of the rod is

$$\theta=\frac{M_t}{GI_p}. \tag{6.3}$$

The product GI_p in the denominator is termed the *torsional rigidity*, or *stiffness*.

The total angle of twist is obtained by multiplying θ by the length of the bar l (henceforth $\varphi=\psi$)

$$\varphi=\frac{M_t l}{GI_p}. \tag{6.4}$$

This formula indicates that the total angle of twist of the bar is directly proportional to the torque M_t, the length of the bar l and inversely proportional to the torsional stiffness GI_p.

Of course, it should be remembered that we used Hooke's law in the derivation of formula (6.4); we thereby assume that the magnitude of the torque is such that the stresses do not exceed the elastic limit of the material.

In formula (6.4) the total angle of twist is expressed in radians. The conversion to degrees is made using the well-known formula

$$\varphi^\circ=\varphi\frac{180^\circ}{\pi}; \tag{6.5}$$

consequently,

$$\varphi^{\circ}=\frac{180^{\circ}}{\pi}\frac{M_t l}{GI_p}. \tag{6.6}$$

Substitute the expression for the angle of twist per unit length from formula (6.3) in the equation (b)

$$\tau_{\rho}=G\rho\frac{M_t}{GI_p}=\frac{M_t\rho}{I_p}. \tag{6.7}$$

This equation indicates that the stresses on elementary areas are directly proportional to their distances from the centre of the section (Fig. 75). For $\rho=0$

$$\tau=0,$$

i. e., no stress occurs on the axis at the centres of sections. As one moves away from the centre to the periphery of the bar, the stresses increase according to a linear law.

The maximum stress occurs at the surface of a circular bar in torsion; it is equal to

$$\tau_{\max}=\frac{M_t r}{I_p}. \tag{6.8}$$

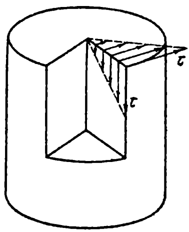


Fig. 75

In distinction to the previously considered types of deformation, the stresses induced by torsion are not uniformly distributed over the section but increase from the centre to the edges of the section.

The diagram showing the variation of stresses along any radius of the section is presented in Fig. 75. By virtue of the law of equal shearing stresses the latter occur on longitudinal sections as well. A shaft of fibrous material (wood, for example) having a lower resistance to shear parallel to grain than perpendicular to grain and loaded in torsion to rupture exhibits a crack in the longitudinal direction if fibres are parallel to the longitudinal axis.

The interior of a shaft, being the least stressed, is often removed altogether, i. e., the shaft is made hollow. The stresses in a hollow cylinder are only slightly increased as compared with a solid cylinder of the same diameter but the reduction in weight is appreciable. That is why shafts of all aircraft engines for which saving of weight is of paramount importance are drilled out.

Formula (6.8) is usually written in an alternate form. The ratio of the polar moment of inertia I_p to the maximum radius r of the section is called the *polar section modulus* and denoted by Z_p,

i. e.,

$$Z_p = \frac{I_p}{r}. \tag{6.9}$$

In this case formula (6.8) becomes

$$\tau_{max} = \frac{M_t}{Z_p}. \tag{6.10}$$

Formula (6.4), providing a means for determining the deformation, and formula (6.10), expressing the maximum stress, are the basic formulas in the theory of torsion of circular cylinders.

40. Polar Moment of Inertia and Section Modulus of a Circle and a Circular Ring

The design of solid and hollow shafts by formulas (6.4) and (6.10) involves the determination of the polar moment of inertia and section modulus of a circle and a circular ring.

The polar moment of inertia of a figure was defined as the expression $\int\limits_A \rho^2 dA$, i. e., the sum of products of elementary areas dA of the figure and the squares of their distances (ρ^2) to some pole which is extended over the entire area of the figure. Using this definition, let us calculate the polar moment of inertia of a circle. To do this, we imagine a circle (Fig. 76) to be divided into an infinite number of infinitely thin rings. Denote the radius of one of these rings by ρ and its thickness by $d\rho$; the area of the elementary ring can then be expressed as

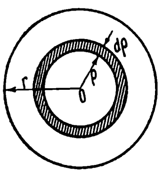


Fig. 76

$$dA = 2\pi\rho d\rho.$$

All points of this elementary ring may be assumed to be removed the same distance ρ from the pole, i. e., from the centre of the circle. Multiplying the area of the ring dA by the square of the distance of its points to the pole (ρ^2) and integrating this product from $\rho = 0$ to $\rho = r$, i. e., summing up the products $\rho^2 dA$ over the entire area of the circle, we obtain

$$I_p = \int\limits_A \rho^2 dA = \int\limits_0^r \rho^2 2\pi\rho \, d\rho = 2\pi \left| \frac{\rho^4}{4} \right|_0^r = \frac{\pi r^4}{2}. \tag{6.11}$$

Substitute $d/2$ for r in formula (6.11). We then obtain an alternate expression for the polar moment of inertia of a circle

$$I_p = \frac{\pi d^4}{32} \cong 0.1 d^4. \tag{6.12}$$

The polar moment of inertia of a circular ring (Fig. 77) of outer diameter D and inner diameter d is found as the difference between the polar moments of inertia of the outer and inner circles

$$I_p=\frac{\pi D^4}{32}-\frac{\pi d^4}{32}=\frac{\pi(D^4-d^4)}{32}\cong 0.1\,(D^4-d^4). \tag{6.13}$$

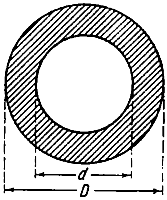


Fig. 77

In the design of hollow shafts, it is convenient to have the expression for the polar moment of inertia of a circular ring in an alternate form.

Denoting the ratio d/D by α, we obtain from expression (6.13)

$$I_p=\frac{\pi D^4}{32}-\frac{\pi d^4}{32}=\frac{\pi}{32}[D^4-(D\alpha)^4]$$

or

$$I_p=\frac{\pi D^4}{32}(1-\alpha^4). \tag{6.14}$$

If a ring is very thin, then, taking into account that $D(1-\alpha)=2\delta$, where δ is the wall thickness, $1+\alpha\cong 2$ and $1+\alpha^2\cong 2$ we obtain

$$I'_p=\frac{\pi D^3}{8}\delta. \tag{6.14'}$$

The polar section modulus for a circle is

$$Z_p=\frac{I_p}{r}=\frac{\pi}{2}\frac{r^4}{r}=\frac{\pi r^3}{2}$$

or

$$Z_p=\frac{\pi d^3}{16}\cong 0.2d^3. \tag{6.15}$$

The polar section modulus for a circular ring is

$$Z_p=\frac{I_p}{\frac{D}{2}}=\frac{\pi(D^4-d^4)}{32\frac{D}{2}}=\frac{\pi}{16}\frac{(D^4-d^4)}{D}\cong 0.2\frac{D^4-d^4}{D} \tag{6.16}$$

or

$$Z_p=\frac{\pi}{16}D^3(1-\alpha^4)\cong 0.2D^3(1-\alpha^4). \tag{6.17}$$

For a very thin ring we obtain from (6.14')

$$Z_p=\frac{\pi D^2}{4}\delta. \tag{6.17'}$$

Note that the section modulus of a circular ring cannot be determined as the difference between the section moduli of the large and small circles.

41. Design Equations in Torsion

Denoting the allowable stress in torsion by $[\tau]$, just as the allowable stress in shear, we obtain the strength-design formula in torsion

$$\tau_{\max}=\frac{M_t}{Z_p}\leqslant[\tau]. \tag{6.18}$$

The value of the allowable stress $[\tau]$ is taken as 0.5 to 0.6 of the allowable stress in tension.

Assigning a factor of safety k, the allowable stress for a ductile material is determined from the yield strength in torsion

$$[\tau]=\frac{\tau_y}{k}, \tag{6.19}$$

where for steel τ_y is approximately equal to 0.5 to 0.6 of σ_y. The design equation (6.18) is similar in form to the design equations for tension, compression and shear. Here, too, the internal force factor (moment and not a force) is divided by a certain geometric factor characterizing the section, viz. by the section modulus (and not by the cross-sectional area as is done for tension, compression and shear).

The design equation (6.18) is valid for circular (both solid and hollow) shafts. For any other sections it cannot be applied.

Equation (6.18) involves three quantities: M_t, Z_p and $[\tau]$. Consequently, knowing any two of these quantities, it is easy to determine the third one.

In the analysis of an operating shaft when M_t and Z_p are known, the stress is determined by formula (6.10).

Having the shaft dimensions and knowing the allowable stress for its material under given operating conditions, we can determine the maximum permissible torque that can be transmitted by the shaft, using formula (6.18),

$$M_t\leqslant[\tau]\,Z_p. \tag{6.20}$$

In designing a new shaft it is necessary to know the allowable stress and the torque in order to determine shaft dimensions. Thus, substituting the expression for the section modulus in formula (6.18), we obtain for a solid shaft

$$\frac{16M_t}{\pi d^3}\leqslant[\tau], \tag{6.21}$$

whence

$$d \geqslant \sqrt[3]{\frac{16}{\pi}\frac{M_t}{[\tau]}} = 1.72\sqrt[3]{\frac{M_t}{[\tau]}}. \tag{6.22}$$

Substituting the expression for the torque in terms of H and n (6.1) in this formula, we obtain

$$d \geqslant 1.72\sqrt[3]{\frac{71{,}620\,H}{n\,[\tau]}} = 72\sqrt[3]{\frac{H}{n\,[\tau]}}. \tag{6.23}$$

Similarly, we find for hollow shafts

$$D \geqslant 1.72\sqrt[3]{\frac{M_t}{(1-\alpha^4)\,[\tau]}} \tag{6.24}$$

or

$$D \geqslant 1.72\sqrt[3]{\frac{71{,}620\,H}{n\,[\tau]}} = 72\sqrt[3]{\frac{H}{n\,(1-\alpha^4)\,[\tau]}}. \tag{6.25}$$

From formulas (6.23) and (6.25) it is seen that, for a given power, any increase in the number of revolutions n results in a reduction in the shaft diameter.

In practice, in addition to the strength requirement, the stiffness condition is usually imposed, namely, that the angle of twist per unit length of the shaft must not exceed a specified value. This is done to avoid excessive deformations of the shaft. The value of the permissible angle of twist depends on the case at hand.

The smaller the angle of twist, the less is the flexibility of the shaft. In some cases high flexibility of the shaft is desirable. For instance, in transmitting a variable torque from one shaft to another an intermediate shaft of low stiffness which acts as a spring is placed to smooth off the irregular action of torques.

To obtain the stiffness-design formula, we introduce the allowable angle of twist φ°_{al} per metre in expression (6.6)

$$\varphi^\circ = \frac{180^\circ M_t 100}{\pi G I_p} \leqslant \varphi^\circ_{al},$$

whence

$$I_p \geqslant \frac{18{,}000 M_t}{G\pi\varphi^\circ_{al}}. \tag{6.26}$$

For a solid shaft we obtain

$$\frac{\pi}{32}d^4 \geqslant \frac{18{,}000 M_t}{G\pi\varphi^\circ_{al}}.$$

Consequently, when the shaft is designed on the basis of its stiffness the shaft diameter must satisfy the condition

$$d \geqslant \sqrt[4]{\frac{18{,}000 M_t\, 32}{G\pi^2\varphi^\circ_{al}}} = 15.3\sqrt[4]{\frac{M_t}{G\varphi^\circ_{al}}}. \tag{6.27}$$

Substituting the value of M_t, we obtain

$$d \geqslant 15.3 \sqrt[4]{\frac{71{,}620\,H}{nG\overset{\circ}{\varphi}_{al}}} = 250 \sqrt[4]{\frac{H}{nG\overset{\circ}{\varphi}_{al}}}. \tag{6.28}$$

Similarly, for hollow shafts we have

$$D \geqslant \sqrt[4]{\frac{18{,}000 M_t\, 32}{G\pi^2 (1-\alpha^4)\, \overset{\circ}{\varphi}_{al}}} = 15.3 \sqrt[4]{\frac{M_t}{G(1-\alpha^4)\, \overset{\circ}{\varphi}_{al}}} \tag{6.29}$$

or

$$D = 15.3 \sqrt[4]{\frac{71{,}620\,H}{nG(1-\alpha^4)\, \overset{\circ}{\varphi}_{al}}} = 250 \sqrt[4]{\frac{H}{nG(1-\alpha^4)\, \overset{\circ}{\varphi}_{al}}}. \tag{6.30}$$

If, in the design of a shaft, only the allowable stress is assigned, i. e., the shaft diameter is determined on the basis of its strength, then formula (6.23) is used for solid shafts and formula (6.25) for hollow shafts.

In addition to torsion, driving shafts are subjected to bending produced by the weight of attached parts, by the forces acting on gear teeth, etc. For the sake of simplicity, driving shafts are designed only for torsion but the allowable stress is then reduced.

For shafts of carbon steel, it is customary to take $[\tau] = 120$ to 250 kgf/cm², for shafts of special steel the allowable stress is increased.

If, in the design of a shaft, the allowable angle of twist is assigned as well as the allowable stress, the shaft diameter must also be determined on the basis of its stiffness by formula (6.28) [or (6.30) for hollow shafts]. In order for the shaft to satisfy both requirements (strength and stiffness) simultaneously, the larger of the two values should be used for the diameter.

For shafts, the allowable angle of twist $\overset{\circ}{\varphi}_{al}$ per metre is taken as 0.3° and larger (up to 2°).

Example 34. A shaft is twisted by a torque $M_t = 200$ kgf-m. Determine the shaft diameter if the allowable stress is $[\tau] = 500$ kgf/cm².

Solution. The section modulus of the shaft is, from formula (6.18),

$$Z_p = \frac{M_t}{[\tau]} = \frac{20{,}000}{500} = 40 \text{ cm}^3.$$

Since

$$Z_p = \frac{\pi}{16} d^3,$$

we have

$$d \geqslant \sqrt[3]{\frac{16 Z_p}{\pi}} = \sqrt[3]{\frac{16 \times 40}{3.14}} = 5.89 \cong 6.0 \text{ cm}.$$

Example 35. A hollow shaft is twisted by a torque $M_t = 500$ kgf-m. The ratio of the inner to the outer diameter is

$$\frac{d}{D} = \alpha = 0.7.$$

The allowable stress is $[\tau] = 600$ kgf/cm². Determine cross-sectional dimensions and reduction in weight as compared with a solid shaft of the same strength.

Solution. The outer diameter of the shaft is, by formula (6.24),

$$D \geqslant 1.72 \sqrt[3]{\frac{M_t}{(1-\alpha^4)[\tau]}} = 1.72 \sqrt[3]{\frac{50{,}000}{(1-0.7^4)\,600}} = 8.22 \text{ cm}.$$

The inner diameter of the shaft is

$$d = D\alpha = 8.22 \times 0.7 \cong 5.75 \text{ cm}.$$

In the case of a solid shaft the diameter must be, according to formula (6.22),

$$d_s \geqslant 1.72 \sqrt[3]{\frac{M_t}{[\tau]}} = 1.72 \sqrt[3]{\frac{50{,}000}{600}} = 7.5 \text{ cm}.$$

The ratio of the weights of the shafts is equal to the ratio of their cross-sectional areas. The cross-sectional area of the hollow shaft is

$$A = \frac{\pi}{4}(D^2 - d^2) = 0.785\,(8.2^2 - 5.75^2) = 26.9 \text{ cm}^2.$$

The cross-sectional area of the solid shaft is

$$A_s = \frac{\pi}{4} d_s^2 = 0.785 \times 7.5^2 = 44.2 \text{ cm}^2.$$

Consequently, the reduction in weight is determined by the ratio

$$\frac{A}{A_s} = \frac{26.9}{44.2} = 0.61,$$

i.e., the hollow shaft weighs 39 per cent less than the solid shaft of the same strength.

Example 36. When a steel bar of diameter $d = 15$ mm and length $l = 200$ mm (Fig. 78) is twisted by a torque $M_t = 600$ kgf-cm applied at its free end, point A on the surface moves to point A_1 along a circular arc

$$s = \overset{\frown}{AA_1} = 0.22 \text{ mm}.$$

Determine (1) the maximum torsional stress τ_{max}, (2) the angle of rotation φ of the free end of the bar relative to the fixed end, (3) the shearing strain at the surface of the bar, and (4) the modulus of elasticity G of the bar material.

Solution. (1) To determine the maximum torsional stress we first calculate the polar section modulus using formula (6.15)

$$Z_p = \frac{\pi}{16} d^3 = \frac{\pi}{16} 1.5^3 = 0.663 \text{ cm}^3.$$

The maximum torsional stress is determined by formula (6.10)

$$\tau_{max} = \frac{M_t}{Z_p} = \frac{600}{0.663} = 906 \text{ kgf/cm}^2.$$

(2) Determine the angle of rotation φ of the free end of the bar relative to the fixed end. To do this, we express the arc length s

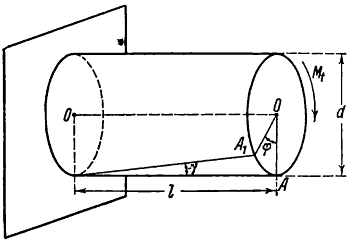


Fig. 78

in terms of the angle φ and the radius of the bar $d/2$

$$s = \varphi \frac{d}{2},$$

whence

$$\varphi = \frac{2s}{d} = \frac{2 \times 0.022}{1.5} = 0.0293 \text{ radian}.$$

(3) The shearing strain γ at the surface of the bar is

$$\gamma = \frac{\smile{AA_1}}{l} = \frac{s}{l} = \frac{0.022}{20} = 0.0011.$$

(4) The modulus of elasticity G of the bar material is, from formula (6.4),

$$G = \frac{M_t l}{I_p \varphi}.$$

The polar moment of inertia of the section of the bar is

$$I_p = \frac{\pi}{32} d^4 = \frac{\pi}{32} \times 1.5^4 = 0.497 \text{ cm}^4.$$

Consequently,

$$G=\frac{600\times 20}{0.497\times 0.0293}=824{,}000\ \text{kgf/cm}^2.$$

The solution of this example can be verified by determining τ_{max} in terms of γ and G. According to Hooke's law we have

$$\tau_{max}=G\gamma=824{,}000\times 0.0011=906\ \text{kgf/cm}^2.$$

This stress agrees with the previously calculated value.

Example 37. Two shafts of the same length l, one of which is hollow, transmit equal torques M_t and have equal maximum stresses τ_{max}. Determine the ratio between the angles of twist of these shafts.

Solution. Denote the angle of twist of the solid shaft by φ and that of the hollow shaft by φ'; the polar moments of inertia of the sections of the shafts are denoted, respectively, by I_p and I_p'. The diameter of the solid shaft is d and the outer diameter of the hollow shaft D.

Since the transmitted torques and the maximum stresses are the same for the two shafts, the polar section moduli Z_p of the shafts must also be the same.

The angle of twist of the solid shaft is, on the basis of formula (6.4),

$$\varphi=\frac{M_t l}{GI_p}.$$

Taking into account that

$$M_t=\tau_{max}Z_p=\tau_{max}\frac{I_p}{\frac{d}{2}},$$

we obtain

$$\varphi=\frac{M_t l}{GI_p}=\frac{\tau_{max}2I_p l}{dGI_p}=\frac{2\tau_{max}l}{dG}.$$

Similarly, we obtain for the hollow shaft

$$\varphi'=\frac{M_t l}{GI_p}=\frac{\tau_{max}2I_p' l}{DGI_p'}=\frac{2\tau_{max}l}{DG}.$$

The ratio of the angles of twist of the shafts is

$$\frac{\varphi}{\varphi'}=\frac{2\tau_{max}l}{dG}:\frac{2\tau_{max}l}{DG}=\frac{D}{d}.$$

Consequently, under the given conditions *the angles of twist are inversely proportional to the diameters.*

Example 38. A shaft (Fig. 71*a*) carries four pulleys, A, B, C and D. Pulley A receives a torque $M_A=100$ kgf-m from the mo-

tor, pulleys B, C and D are transmission ones. Pulley B transmits a moment $M_B = 25$ kgf-m to the machine, pulley C transmits a moment $M_C = 25$ kgf-m and pulley D transmits a moment $M_D = 50$ kgf-m. It is required (1) to plot a twisting moment diagram, (2) to determine the shaft diameter if the allowable stress is $[\tau] = 200$ kgf/cm², (3) to find the angle of twist of the shaft in each portion and (4) to determine the shaft diameter in each portion if the stress is to be equal to the allowable value throughout the length of the shaft. The distances between the pulleys are the same, $l = 1$ m, the modulus of elasticity $G = 8 \times 10^5$ kgf/cm².

Solution. (1) The twisting moment diagram plotted as explained above is represented in Fig. 71*b*.

(2) As is seen from this diagram, the most severely stressed portions of the shaft are AC and AD where the moments are the same and equal to 50 kgf-m. Based on the value of this twisting moment, we determine the shaft diameter using formula (6.22)

$$d = 1.72 \sqrt[3]{\frac{M_t}{[\tau]}} = 1.72 \sqrt[3]{\frac{5{,}000}{200}} = 5.03 \text{ cm}.$$

Round off this value to $d = 5$ cm.

(3) The angle of twist in portion BC is

$$\varphi_{BC} = \frac{M_t l}{G I_p} = \frac{2{,}500 \times 100}{8 \times 10^5 \times \frac{\pi}{32} \times 5^4} = 0.0051 \text{ radian}.$$

The angles of twist in portions CA and AD are

$$\varphi_{CA} = \varphi_{AD} = \frac{5{,}000 \times 100}{8 \times 10^5 \times \frac{\pi}{32} \times 5^4} = 0.0102 \text{ radian}.$$

The angle of rotation of section B relative to section A is equal to the sum of the angles of twist φ_{BC} and φ_{CA}, i. e.,

$$\varphi_{BA} = \varphi_{BC} + \varphi_{CA} = 0.0051 + 0.0102 = 0.0153 \text{ radian}$$

or in degrees

$$\varphi^\circ = \frac{180}{\pi} \varphi_{BA} = 0.885^\circ.$$

(4) Introducing now the allowable stress, we find that the shaft diameter in portions CA and AD must remain equal to 5 cm while in portion BC it is to be taken as

$$d = 1.72 \sqrt[3]{\frac{2{,}500}{200}} = 3.96 \text{ cm}$$

or, rounding off, $d = 4$ cm.

Example 39. Determine the power transmitted by an engine shaft if the angle of twist of the shaft measured in a length of

1.5 m is 0.5°. The shaft diameter is $d=100$ mm, the number of revolutions $n=500$ rpm, the modulus of elasticity of the material $G=8\times10^5$ kgf/cm².

Solution. From formula (6.4) we have

$$M_t=\frac{\varphi G I_p}{l}.$$

Substituting this value of the twisting moment in formula (6.1), we obtain

$$\frac{\varphi G I_p}{l}=71{,}620\frac{H}{n},$$

whence

$$H=\frac{\varphi G I_p n}{71{,}620 l}.$$

The angle of twist of the shaft expressed in radians is

$$\varphi=\frac{\varphi^\circ\pi}{180}=\frac{0.5\times3.14}{180}.$$

Substituting the numerical values on the right-hand side of the expression for the power H, we have

$$H=\frac{0.5\times3.14\times8\times10^5\times3.14\times10^4\times500}{180\times71{,}620\times150\times32}=318 \text{ hp}.$$

Example 40. A steel shaft of diameter $d=8$ cm is fixed at both ends (Fig. 79*a*). At an intermediate section a distance $a=0.5$ m from the left end and a distance $b=1$ m from the right end, a torque $M_t=750$ kgf-m is applied. Determine the stress in the shaft and the angle of twist.

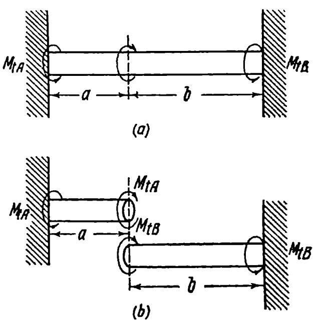


Fig. 79

Solution. We first determine the reactive moments at the fixed ends, M_{tA} and M_{tB}.

The equilibrium conditions of statics give only one equation for their determination (Fig. 79*b*)

$$M_{tA}+M_{tB}=M_t. \quad \text{(a)}$$

Consequently, this problem is a statically indeterminate one.

The second equation necessary for the solution of the problem is set up by considering the deformation of the shaft. The angle of rotation of the section at

which the torque is applied can be determined in two ways, namely,

$$\varphi=\frac{M_{tA}a}{GI_p} \quad \text{and} \quad \varphi=\frac{M_{tB}b}{GI_p}.$$

Consequently,

$$\frac{M_{tA}a}{GI_p}=\frac{M_{tB}b}{GI_p}$$

or

$$\frac{M_{tA}}{M_{tB}}=\frac{b}{a}, \tag{b}$$

i. e., the reactive moments are inversely proportional to the distances a and b from the section at which the torque is applied.

From the equations (a) and (b) we find

$$M_{tA}=M_t\frac{b}{a+b} \quad \text{and} \quad M_{tB}=M_t\frac{a}{a+b}.$$

Substituting the values of M_t, a and b, we obtain

$$M_{tA}=750\frac{1}{0.5+1}=500\ \text{kgf-m}, \qquad M_{tB}=750\frac{0.5}{0.5+1}=250\ \text{kgf/m}.$$

Clearly, the larger stress occurs in the left portion of the shaft

$$\tau=\frac{M_{tA}}{Z_p}=\frac{50{,}000}{\frac{\pi}{16}\times 8^3}=500\ \text{kgf/cm}^2.$$

The total angle of twist is

$$\varphi=\frac{M_{tA}a}{GI_p}=\frac{50{,}000\times 50}{8\times 10^5\frac{\pi}{32}\times 8^4}=\frac{1}{128}\ \text{radian}$$

or

$$\varphi^\circ=\frac{180}{\pi}\frac{1}{128}\cong 0.45^\circ.$$

42. Elements of Design of Bars of Rectangular Section for Torsional Loads

As mentioned above, the torsion formulas derived in the preceding sections are valid only for circular bars. The derivation of these formulas is based on the assumption that plane cross sections of a bar remain plane during twisting. Experimental observations support this assumption only for circular bars. When bars of non-circular section are twisted, the cross sections warp. A simple torsion test made with a rubber bar of rectangular section demonstrates this. Figure 80*a* shows the faces of a rubber bar of rectangular section with a network of mutually perpendicular lines

marked on its surface. As is seen from this figure, the transverse lines become curved after twisting. The distortions are greatest at the middle of the sides of the cross section. At the corners there is no distortion at all. Because of the warping of cross sections the torsion theory for bars of rectangular section becomes complicated. Figure 80*b* shows the torsional stress diagram for a rectangular section.

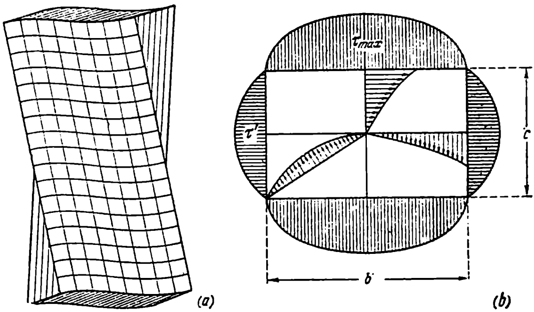


Fig. 80

The maximum stress occurring at the middle of the longer side of the cross section is determined by the formula

$$\tau_{\max}=\frac{M_t}{\alpha bc^2}, \tag{6.31}$$

where b is the longer side and c the shorter side of the cross section, α is a numerical factor depending on the ratio b/c. The values of this factor for various values of b/c are given in Table 9.

Table 9. Factors α and β

b/c	1	1.5	2	3	4	6	8	10	∞
α	0.208	0.231	0.246	0.267	0.282	0.299	0.307	0.313	0.333
β	0.141	0.196	0.229	0.263	0.281	0.299	0.307	0.313	0.333
γ	1.000	0.858	0.796	0.753	0.745	0.743	0.743	0.743	0.743

The stress can also be determined using the following approximate formula

$$\tau_{max} = \frac{M_t}{bc^2}\left(3 + 1.8\frac{c}{b}\right). \tag{6.32}$$

The maximum stress τ' at the middle of the shorter side of the cross section is determined by the formula

$$\tau' = \gamma\tau_{max}. \tag{6.33}$$

The angle of twist is given by

$$\varphi = \frac{M_t l}{\beta bc^3 G}, \tag{6.34}$$

where γ, β are numerical factors depending, just as the factor α, on the ratio b/c. The values of β are given in Table 9.

Example 41. A steel bar of length $l = 4$ m and rectangular section of sides $b = 17$ cm and $c = 10$ cm is twisted by a torque $M_t = 250{,}000$ kgf-cm. Determine the maximum stress and the angle of twist assuming $G = 8 \times 10^5$ kgf/cm².

Solution. The maximum stress at the middle of the longer side b is determined by formula (6.31)

$$\tau_{max} = \frac{M_t}{\alpha bc^2}.$$

To determine α from Table 9 we find the ratio of the sides of the rectangular section

$$\frac{b}{c} = \frac{17}{10} = 1.7.$$

In Table 9 no value of the factor α is available for $b/c = 1.7$; there are values for $b/c = 1.5$ and for $b/c = 2$; the value of α for $b/c = 1.7$ is determined by interpolation

$$\alpha = 0.231 + \frac{(0.246 - 0.231)\,0.2}{0.5} = 0.231 + 0.006 = 0.237,$$

$$\tau_{max} = \frac{250{,}000}{0.237 \times 17 \times 10^2} = 620 \text{ kgf/cm}^2.$$

Determine the same stress by the approximate formula (6.32)

$$\tau_{max} = \frac{250{,}000}{17 \times 10^2}\left(3 + 1.8\,\frac{10}{17}\right) = 595 \text{ kgf/cm}^2.$$

The difference in the stresses is about 4 per cent. The angle of twist is determined by formula (6.34)

$$\varphi = \frac{M_t l}{\beta bc^3 G}.$$

The value of the factor β is found from Table 9 by interpolation

$$\beta = 0.196 + \frac{(0.229-0.196)\,0.2}{0.5} = 0.207.$$

Consequently, the angle of twist is

$$\varphi = \frac{250{,}000\times 400}{0.207\times 17\times 10^3\times 8\times 10^5} = 0.0358 \text{ radian}$$

or

$$\varphi^\circ = \frac{180}{\pi}\times 0.0358 = 2.05^\circ.$$

43. Potential Energy in Torsion

If a cylindrical bar is twisted by a torque within the elastic limit, the work done by the couple is stored in the bar as potential energy. After the torque ceases to act the bar will untwist and give up the stored energy. *Within the elastic limit, the angle of twist is proportional to the twisting moment* (Hooke's law). If twisting moments M_t, increasing from zero, are plotted as ordinates and the corresponding angles of twist φ as abscissas, the M_t-φ relation is represented by a straight line OA (Fig. 81). Let an angle φ_1 correspond to an intermediate value of twisting moment M_1. If the twisting moment is increased by an infinitesimal amount dM_1, the angle φ_1 receives the corresponding increment $d\varphi_1$; the work done is equal to the product $\left(M_1+\frac{dM_1}{2}\right)d\varphi_1$ and is represented graphically by the shaded trapezoid. As the twisting moment increases from zero to a final value M_t, the total strain energy equal to the stored potential energy is represented by the area of triangle OAB and is given by

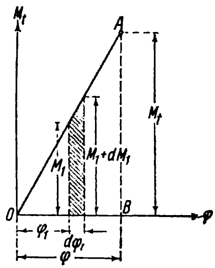


Fig. 81

$$W = \frac{M_t\varphi}{2}. \tag{6.35}$$

Substituting the value of the angle φ from formula (6.4), we obtain

$$W = \frac{M_tM_tl}{2GI_p} = \frac{M_t^2 l}{2GI_p}. \tag{6.36}$$

Example 42. Determine the potential energy stored in a steel bar of diameter $d=12$ mm and length $l=400$ mm when it is twisted by a torque $M_t=200$ kgf-cm; $G=8\times 10^5$ kgf/cm^2.

Solution. Substituting the given values in formula (6.36), we obtain

$$W=\frac{M_t^2 l}{2GI_p}=\frac{200^2\times 40}{2\times 8\times 10^5\times\frac{3.14\times 1.2^4}{32}}=4.92 \text{ cm-kgf.}$$

44. Design of Closely Coiled Helical Springs

Consider a closely coiled helical spring made of round wire and stretched by axial forces P (Fig. 82*a*). Since the pitch is small, we assume that the planes of coils of the spring are perpendicular

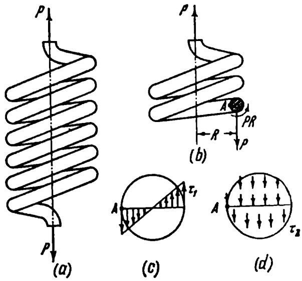


Fig. 82

to the spring axis. Cut a coil of the spring by a plane passing through the spring axis. Remove one part of the spring and consider the equilibrium of the remaining part (Fig. 82*b*). To maintain equilibrium, it is necessary to apply at the centre of the section a downward force P parallel to the spring axis and a moment PR, where R is the mean radius of the spring. Since the moment PR acts in the plane of the section, it produces torsional stresses on the section (Fig. 82*c*) the maximum value of which in the outer fibres is

$$\tau_1=\frac{PR}{Z_p}=\frac{16PR}{\pi d^3},$$

where d is the diameter of the cross section of the wire. The force P acting in the plane of the cross section induces shearing stresses which are assumed to be uniformly distributed over the section (Fig. 82*d*). These stresses are given by

$$\tau_2=\frac{P}{A}=\frac{4P}{\pi d^2}.$$

The total stress in the outer fibres of the spring wire is found by adding geometrically the stresses τ_1 and τ_2. The maximum stress occurs at a point on the periphery of the section at which the directions of the stresses τ_1 and τ_2 coincide. It is clear that this point is A. The stress at this point is

$$\tau_{max} = \tau_1 + \tau_2 = \frac{16PR}{\pi d^3} + \frac{4P}{\pi d^2} = \frac{16PR}{\pi d^3}\left(1 + \frac{d}{4R}\right). \qquad (6.37)$$

We have considered a spring subjected to tension; the same result is obtained when a spring is in compression. In the design of springs in which the mean radius R of a spring exceeds many times the diameter d of the wire from which it is made, the second term in the parentheses is usually neglected. For such springs, formula (6.37) is simplified and takes the form

$$\tau_{max} = \frac{16PR}{\pi d^3}, \qquad (6.38)$$

i. e., such springs are designed only for torsion. The design formula for springs is

$$\tau_{max} = \frac{16PR}{\pi d^3} \leqslant [\tau], \qquad (6.39)$$

where $[\tau]$ is the allowable stress. The allowable stress $[\tau]$ for steels of which springs are made is taken as 3,000 to 7,000 kgf/cm².

In addition to the design for strength it is often necessary to determine the extension or compression (deflection) of a spring, i. e., its deformation.

In determining the deformation of a spring it is customary to consider only the twisting of coils.

Let the upper end of a spring be fixed and the lower end be acted on by a tensile force P. Suppose that this force increases from zero to its final value P. If the lower end of the spring moves downward an amount f, the work done by the force is $Pf/2$.

But this work is equal to the potential energy stored in the spring as a result of twisting of coils. Consequently, on the basis of formula (6.36) we can write the equality

$$\frac{Pf}{2} = \frac{M_t^2 l}{2GI_p}. \qquad (a)$$

Here l is the length of the straightened spring

$$l = 2\pi Rn,$$

where n is the number of coils, $M_t = PR$, $I_p = \pi d^4/32$. Substituting these values in the expression (a), we obtain

$$\frac{Pf}{2} = \frac{P^2R^2\,2\pi Rn\,32}{2G\pi d^4},$$

whence the deflection of the spring is

$$f=\frac{64PR^3n}{Gd^4}. \tag{6.40}$$

Formulas (6.37) through (6.40) derived for the design of tension springs are also valid for compression springs.

Example 43. Determine the maximum stress and the deflection of a helical spring if the compressive force is $P=60$ kgf, the mean radius of the spring $R=25$ mm, the wire diameter $d=6$ mm, the number of effective coils $n=5$ and $G=825{,}000$ kgf/cm².

Solution. By formula (6.38), the maximum stress is

$$\tau_{max}=\frac{16PR}{\pi d^3}=\frac{16\times60\times2.5}{3.14\times0.6^3}=3{,}540 \text{ kgf/cm}^2$$

and by formula (6.40) the deflection of the spring is

$$f=\frac{64PR^3n}{Gd^4}=\frac{64\times60\times2.5^3\times5}{825{,}000\times0.6^4}=2.8 \text{ cm}.$$

Example 44. Determine the force required to produce a deformation of 1 mm per turn if the mean radius of the spring is $R=35$ mm, the wire diameter $d=8$ mm, the number of effective coils $n=1$ and $G=825{,}000$ kgf/cm².

Solution. From formula (6.40), the force P is

$$P=\frac{fGd^4}{64R^3n}.$$

Substituting $f=0.1$ cm, $n=1$ and the values of the remaining quantities, we obtain

$$P=\frac{0.1\times825{,}000\times0.8^4}{64\times3.5^3}=12.3 \text{ kgf}.$$

45. Design of Shafts Based on Allowable Loads

The foregoing analysis of twisted shafts based on the strength condition implied that the maximum shearing stress at the surface of the shaft was not greater than the allowable stress $[\tau]$. We assumed that, for a shaft of ductile material (Fig. 83 *a*), failure occurred when the maximum stress at the surface of the shaft reached the yield point stress τ_y, and required that $[\tau]=\tau_y/k$, where k is a factor of safety.

At the same time it is known that, for a ductile material exhibiting a considerable plastic elongation at the yield point, an increase in the torque, after the maximum stress reaches τ_y, causes no further increase of stresses in the extreme fibres of the shaft due to yielding of the material. However, the stresses in the elastic region of the section will grow until the stresses become equal to τ_y

throughout the section. This state of stress is shown in Fig. 83*b*. In this limiting state the load-carrying capacity of the shaft is exhausted.

Calculate the magnitude of the torque for this limiting state. Divide the cross-sectional area of the shaft into infinitely thin an-

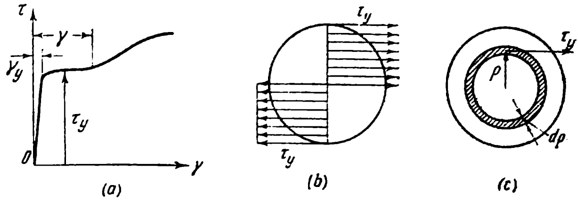


Fig. 83

nular areas (Fig. 83*c*). The elementary twisting moment due to the internal forces on the area dA a distance ρ from the centre is

$$dM = \tau_y \rho \, dA = \tau_y \rho 2\pi\rho \, d\rho.$$

From the condition of equilibrium of external and internal moments we find the limiting external moment

$$M_{\mathrm{lim}} = \int_0^{\frac{d}{2}} \tau_y \rho \, 2\pi\rho \, d\rho = \frac{1}{12} \pi d^3 \tau_y.$$

Using a factor of safety k, the allowable torque is

$$[M_t] = \frac{M_{\mathrm{lim}}}{k} = \frac{\pi d^3 \tau_y}{12k} = \frac{\pi d^3}{12} [\tau],$$

whence

$$d \geqslant \sqrt[3]{\frac{12 [M_t]}{\pi [\tau]}}. \qquad (6.41)$$

According to the conventional design based on the allowable stress [formula (6.22)]

$$d \geqslant \sqrt[3]{\frac{16 M_t}{\pi [\tau]}}. \qquad (6.22)$$

It follows that, for the same factor of safety, the shaft diameter determined from the allowable load is $\sqrt[3]{12/16} = 0.91$ of the diameter determined from the allowable stress.

Note that the design based on the allowable load, which provides a saving in material, applies only to shafts of ductile material transmitting a constant torque when the strength criterion is the yield point of the material.

46. Check Questions

What is the relation between the torque, the power transmitted by a shaft and the number of revolutions of the shaft?

What are the assumptions underlying the theory of torsion of circular bars?

What is the total angle of twist?

What is the polar moment of inertia? Its dimension?

Write the formula for the total angle of twist.

Define the torsional stiffness of a bar.

At what points do the maximum stresses occur in a shaft subjected to torsion?

How are the torsional stresses distributed over the cross section of a shaft in torsion?

Using the law of equal shearing stresses (4.7), find the magnitude and direction of σ_{max} and σ_{min} in Fig. 75.

What is gained by drilling out shafts?

What formula gives the maximum torsional stress in a bar of circular section under torsional loading?

What is the polar section modulus? Its dimension?

What formulas define the polar moment of inertia and the polar section modulus of a circle and a circular ring?

How must the shaft diameter be changed if the power transmitted remains unchanged and the number of revolutions increases?

At what points of the section do the maximum stresses occur in a rod of rectangular section under torsional loading?

What is the expression for strain energy in torsion?

What stresses occur in coils of a helical spring under compression and tension?

What considerations are used and what formula is applied in determining the deflection of a spiral spring?

Chapter VII

Static moments, centroids and moments of inertia of plane figures

47. Static Moments of Plane Figures

In the analysis of the torsion of circular rods we encountered the integral $\int_A \rho^2 \, dA$, which was termed the polar moment of inertia. Definite integrals of this kind called axial moments of inertia will be encountered in the study of bending.

Moments of inertia are geometric quantities. In the case of torsion and bending they play much the same part as cross-sectional areas for tension and compression.

The present chapter provides fundamental information on moments of inertia of plane figures necessary in the further study of strength of materials. The determination of moments of inertia is closely related to the determination of static moments and centroids of areas. Therefore, at the beginning of this chapter we shall recall the procedures for calculating static moments already known from theoretical mechanics.

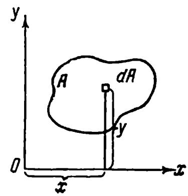


Fig. 84

The *static moment* of an area with respect to any axis x (Fig. 84) taken in the same plane is defined as the sum of the products of elementary areas dA and their distances to the axis; this sum is extended over the whole area A.

The static moment with respect to the x axis is

$$S_x = \int_A y \, dA. \tag{7.1}$$

The static moment with respect to the y axis is

$$S_y = \int_A x \, dA. \tag{7.2}$$

Regarding an elementary area as a force and its distance from the axis as the arm of the force, we may write, on the basis of the theorem that the sum of moments of components is equal to the

moment of the resultant,

$$\int_A y\,dA = Ay_c, \quad \int_A x\,dA = Ax_c,$$

where x_c and y_c are the co-ordinates of the centroid of the whole area A. Consequently, *the static moment of an area A with respect to any axis is equal to the product of the whole area and the distance of its centroid from this axis*

$$\left.\begin{aligned} S_x &= Ay_c, \\ S_y &= Ax_c. \end{aligned}\right\} \tag{7.3}$$

Since the area A has the dimension cm^2 and the distance of the centroid from the axis (x_c and y_c) has the dimension cm, the static moment will have the dimension cm^3 ($cm \times cm^2 = cm^3$). A static moment may be positive or negative because an area is always a positive quantity and distances x_c and y_c may be positive or negative. *If the axis with respect to which the static moment is being determined passes through the centroid of the area, the static moment with respect to this axis is zero.*

Indeed, from Eqs. (7.3) we obtain for $x_c = 0$ and $y_c = 0$

$$S_x = A \times 0 = 0, \qquad S_y = A \times 0 = 0.$$

If a figure has an axis of symmetry, the latter always passes through the centroid of the figure and hence the static moment of the area with respect to an axis of symmetry is always zero.

If a composite figure can be broken into simple figures whose areas and centroids are easy to determine, the static moment of the entire figure with respect to any axis can be found as the sum of the static moments of its individual parts with respect to the same axis

$$S_x = S_{1x} + S_{2x} + S_{3x} + \ldots + S_{nx}, \tag{7.4}$$

where S_x is the static moment of the entire figure and S_{1x}, S_{2x}, S_{3x}, ..., S_{nx} are the static moments of the individual parts of the figure.

If the areas of the individual parts of a composite figure are denoted by A_1, A_2, A_3, ..., A_n and the distances of their centroids from the x axis by $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$, ..., $\bar{y}_n$, expression (7.4) may be rewritten as

$$(A_1 + A_2 + A_3 + \ldots + A_n)\,y_c = A_1\bar{y}_1 + A_2\bar{y}_2 + A_3\bar{y}_3 + \ldots + A_n\bar{y}_n,$$

whence the distance of the centroid of the entire figure from the x axis is

$$y_c = \frac{A_1\bar{y}_1 + A_2\bar{y}_2 + A_3\bar{y}_3 + \ldots + A_n\bar{y}_n}{A_1 + A_2 + A_3 + \ldots + A_n}. \tag{7.5}$$

Similarly, the distance of the centroid of the figure from the y axis may be expressed as

$$x_c = \frac{A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3 + \ldots + A_n\bar{x}_n}{A_1 + A_2 + A_3 + \ldots + A_n}. \qquad (7.6)$$

Example 45. Determine the co-ordinates of the centroid of a triangle of base l and height h (Fig. 85).

Solution. Dropping a perpendicular from vertex B of triangle ABC on to the base AC, we break the triangle into two right triangles the positions of whose centroids are known. Denote base

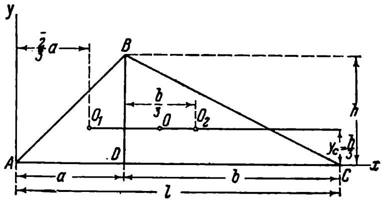


Fig. 85

AD of the left-hand triangle by a and base DC of the right-hand triangle by b.

The centroids O_1 and O_2 of triangles ADB and BDC are, as is known, at one-third of their height ($h/3$) from the base, consequently, the ordinate of the centroid O of the entire triangle ABC will also be at one-third of the height from the base, i. e.,

$$y_c = \frac{h}{3}.$$

Further, the centroid O_1 of the left-hand triangle is at a distance $a/3$ from the vertical leg BD, therefore its distance to the axis of ordinates Ay is $\frac{2}{3}a$.

In the right-hand triangle the centroid O_2 is at a distance $b/3$ from leg BD, consequently, its distance to the axis Ay is $(a + b/3)$.

The static moments of two right triangles with respect to the axis Ay are

triangle ADB

$$S_{1y} = \frac{ah}{2}\frac{2}{3}a = \frac{a^2h}{3},$$

triangle BDC

$$S_{2y} = \frac{bh}{2}\left(a + \frac{b}{3}\right) = \frac{bh(3a+b)}{6}.$$

The distance of the centroid of the entire triangle to the axis Ay is, according to formula (7.6),

$$x_c = \frac{S_{1y}+S_{2y}}{A} = \frac{\frac{a^2h}{3}+\frac{bh(3a+b)}{6}}{\frac{lh}{2}} = \frac{2a^2+3ab+b^2}{3l}.$$

This expression will be easier to remember if it is somewhat rearranged

$$x_c = \frac{2a^2+3ab+b^2}{3l} = \frac{(a^2+2ab+b^2)+a(a+b)}{3l};$$

since $a+b=l$

$$x_c = \frac{l+a}{3}.$$

48. Moments of Inertia of Plane Figures

The *axial (equatorial) moment of inertia* of an area with respect to any axis (Fig. 86) lying in its plane is defined as the sum of the products of elementary areas and the squares of their distances to this axis

$$\left.\begin{aligned} I_x &= \int_A y^2\,dA, \\ I_y &= \int_A x^2\,dA. \end{aligned}\right\} \qquad (7.7)$$

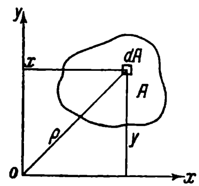


Fig. 86

It is easy to see that the sum of the axial moments of inertia of a plane section with respect to two mutually perpendicular axes is equal to the polar moment of inertia with respect to a pole which is the point of intersection of these axes.

Indeed, joining dA to the origin, we have, by the Pythagorean theorem,

$$\rho^2 = x^2 + y^2;$$

consequently,

$$I_p = \int_A \rho^2\,dA = \int_A (x^2+y^2)\,dA = \int_A x^2\,dA + \int_A y^2\,dA$$

or

$$I_y + I_x = I_p. \qquad (7.8)$$

Formula (7.8) holds for any two mutually perpendicular axes. Consequently, *for all possible rotations of the axes about the origin of co-ordinates the sum of axial moments of inertia remains constant and equal to the polar moment of inertia.*

As is seen from formula (7.7), axial moments of inertia are positive and cannot be zero; they are measured in units of length to the fourth power (cm^4).

The *product of inertia* of an area is defined as the sum of the products of elementary areas and their co-ordinates (i. e., their distances to the two co-ordinate axes) extended over the whole area of the section

$$I_{xy} = \int_A xy\,dA. \tag{7.9}$$

A product of inertia has the dimension cm^4 but, in contrast to an axial and a polar moment of inertia, it may be positive, negative and equal to zero. The sign of a product of inertia depends on the signs of terms $xydA$.

The figures encountered in what follows are of simple geometry. The moments of inertia of such figures are usually determined by integration. If the shape of a figure is complex and cannot be broken into simple figures, the moments of inertia of such figures are determined graphically, by approximate integration or by means of special instruments.

49. Transformation Formulas for Moments of Inertia in the Case of Parallel Transfer of Axes

Axes passing through the centroid of a figure are called *centroidal axes* and the moment of inertia of a figure taken with respect to a centroidal axis is called the *centroidal moment of inertia.* Suppose that for some figure (Fig. 87) the x axis is a centroidal axis with respect to which the moment of inertia I_x is known, and it is required to determine the moment of inertia I_{x_1} of the figure with respect to another axis x_1 parallel to the centroidal one and distant a from it.

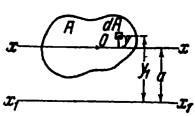


Fig. 87

By definition, the moment of inertia I_x and the unknown moment of inertia I_{x_1} are expressed as follows:

$$I_x = \int_A y^2\,dA, \qquad I_{x_1} = \int_A y_1^2\,dA.$$

From Fig. 87 it is seen that the distances of all elementary areas dA from the new axis x_1 are increased by a constant quantity a, i. e.,

$$y_1 = y + a.$$

Substituting this value of y_1 in the expression for I_{x_1}, we obtain

$$I_{x_1} = \int\limits_A (y+a)^2 dA = \int\limits_A y^2\, dA + 2a \int\limits_A y\, dA + a^2 \int\limits_A dA.$$

The first integral of this expression is the centroidal moment of inertia I_x. The second integral is zero as it represents the static moment of the area with respect to the x axis passing through the centroid of the figure. The third integral is equal to the product a^2A. Consequently,

$$I_{x_1} = I_x + a^2A. \qquad (7.10)$$

This formula, which has a wide practical application, reads as follows: *the moment of inertia of a figure with respect to any axis is equal to the moment of inertia with respect to an axis parallel to*

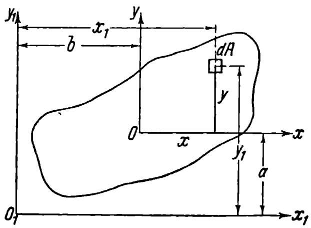


Fig. 88

it and passing through the centroid plus the area of the figure times the square of the distance between the axes.

From formula (7.10) it is seen that of all moments of inertia with respect to parallel axes the least moment is the one with respect to an axis passing through the centroid of the figure, i. e., the centroidal moment of inertia.

Formula (7.10) enables one to determine the centroidal moment of inertia if the moment of inertia with respect to any other axis is known; it may be rewritten as

$$I_x = I_{x_1} - a^2A.$$

Let us derive in the same way a transformation formula for the product of inertia when transferring to parallel axes of co-ordinates.

Let the product of inertia of some figure with respect to its centroidal x and y axes (Fig. 88) be known. It is required to determine the product of inertia of this figure with respect to some other axes, x_1 and y_1, parallel to the centroidal axes.

Denote the distance between the parallel axes by a and b, respectively, as shown in Fig. 88.

The known product of inertia with respect to the centroidal axes is

$$I_{xy}=\int\limits_A xy\,dA.$$

The unknown product of inertia with respect to the x_1 and y_1 axes is

$$I_{x_1y_1}=\int\limits_A x_1y_1\,dA.$$

The new co-ordinates of elementary areas are expressed in terms of the old co-ordinates by the formulas (Fig. 88)

$$x_1=x+b, \qquad y_1=y+a.$$

Substituting these values of x_1 and y_1 in the expression for $I_{x_1y_1}$, we obtain

$$I_{x_1y_1}=\int\limits_A (x+b)(y+a)\,dA=\int\limits_A xy\,dA+b\int\limits_A y\,dA+a\int\limits_A x\,dA+ab\int\limits_A dA.$$

The first integral is I_{xy}, the second and third integrals are equal to zero as they represent the static moments with respect to axes passing through the centroid of the figure. Therefore we finally obtain

$$I_{x_1y_1}=I_{xy}+abA. \tag{7.11}$$

This result is formulated thus: *the product of inertia with respect to arbitrary axes parallel to the centroidal ones is equal to the product of inertia with respect to the centroidal axes plus the area of the figure times the co-ordinates of its centroid with respect to the arbitrary axes.*

50. Moments of Inertia of Some Simple Figures

In this section expressions are derived for axial moments of inertia of simple figures frequently encountered in design practice. A knowledge of moments of inertia of simple figures will help in the determination of moments of inertia of complex figures if their moments of inertia are calculated as the sum of the moments of inertia of the simple figures into which they are broken.

Rectangle. Find the axial moment of inertia of a rectangle of base b and height h with respect to its base (Fig. 89).

Break the area of the rectangle into elementary areas dA of base b and height dy_1; one of these areas is shown as a cross-hatched

strip in the figure

$$dA = b\,dy_1.$$

Substituting this expression for the area of the elementary strip in the general expression for a moment of inertia, we obtain

$$I_{x_1} = \int_A y_1^2\,dA = \int_0^h y_1^2 b\,dy_1 = b\left|\frac{y_1^3}{3}\right|_0^h.$$

The limits of integration 0 and h indicate that the integration is extended over the whole of the rectangle, from the area for which $y_1 = 0$ to the area for which $y_1 = h$. Substituting the limits, we finally find

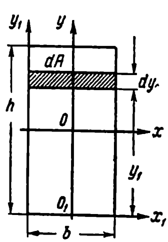


Fig. 89

$$I_{x_1} = \frac{bh^3}{3}. \tag{7.12}$$

By analogy, the moment of inertia of the rectangle with respect to the vertical axis y_1 is written as

$$I_{y_1} = \frac{hb^3}{3}.$$

Let us now calculate the moment of inertia of the rectangle with respect to its centroidal horizontal axis applying formula (7.10)

$$I_x = I_{x_1} - \left(\frac{h}{2}\right)^2 A = \frac{bh^3}{3} - \frac{h^2}{4}\,bh = \frac{bh^3}{12}. \tag{7.13}$$

The moment of inertia of the rectangle with respect to the centroidal vertical axis is

$$I_y = \frac{hb^3}{12}.$$

Square of side a. Substituting $b = h = a$ in formulas (7.12) and (7.13), we obtain

$$I_{x_1} = I_{y_1} = \frac{a^4}{3}, \tag{7.14}$$

$$I_x = I_y = \frac{a^4}{12}. \tag{7.15}$$

Parallelogram (Fig. 90). The axial moments of inertia of a parallelogram with respect to the centroidal x axis and the base are determined by formulas (7.13) and (7.12) derived for a rectangle. This follows from the fact that a parallelogram can be formed from a rectangle by moving elementary areas of infinitesimal height parallel to the base. The moment of inertia of a figure is not

changed if its parts are moved parallel to the axis with respect to which the moment is to be determined since neither elementary areas dA nor their distances from the axis are changed.

Note that the moment of inertia of the parallelogram with respect to the y axis can no longer be calculated by the formula derived for a rectangle since in this case the elementary areas are shifted not parallel to the y axis but perpendicular to it.

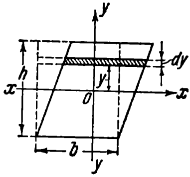


Fig. 90

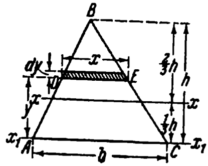


Fig. 91

Triangle. Determine the axial moment of inertia of a triangle of base b and height h with respect to the base and a centroidal x axis parallel to the base (Fig. 91).

Break the triangle into infinitesimal strips by lines parallel to the base, as was done in calculating the moment of inertia of a rectangle. One of these strips is shown by hatching in the figure. The area of this strip a distance y from the base of the triangle is

$$dA = x dy.$$

The length of the strip x is determined from the similar triangles ABC and DBE

$$x = b\frac{h-y}{h},$$

therefore

$$dA = b\frac{h-y}{h}dy.$$

The moment of inertia of the triangle with respect to the base is

$$I_{x_1} = \int_A y^2\, dA = \int_0^h y^2 b\frac{h-y}{h}dy = b\int_0^h y^2 dy - \frac{b}{h}\int_0^h y^3\, dy.$$

Performing the integration, we obtain

$$I_{x_1} = b\left|\frac{y^3}{3}\right|_0^h - \frac{b}{h}\left|\frac{y^4}{4}\right|_0^h = \frac{bh^3}{3} - \frac{bh^3}{4} = \frac{bh^3}{12}. \qquad (7.16)$$

The moment of inertia with respect to the centroidal x axis parallel to the base is determined by formula (7.10)

$$I_x = I_{x_1} - a^2 A.$$

Substituting in this formula the above value of I_{x_1}, the area of the triangle $bh/2$ and the distance from the centroidal axis of the

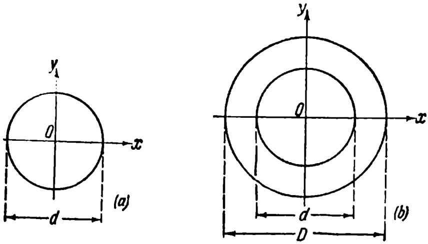


Fig. 92

triangle to the base $h/3$, we obtain

$$I_x = \frac{bh^3}{12} - \left(\frac{h}{3}\right)^2 \frac{bh}{2} = \frac{bh^3}{36}. \tag{7.17}$$

Circle. All centroidal moments of inertia of a circle (Fig. 92*a*) are the same because of the symmetry of the figure, and therefore

$$I_x = I_y = \frac{I_x + I_y}{2}.$$

In Sec. 48 it was proved that the sum of the axial moments of inertia of a figure with respect to two perpendicular axes is equal to the polar moment of inertia with respect to the point of intersection of these axes; consequently,

$$I_x = I_y = \frac{1}{2}(I_x + I_y) = \frac{1}{2} I_p.$$

Substituting the value of the polar moment of inertia of a circle found in Sec. 40, we obtain

$$I_x = I_y = \frac{\pi d^4}{64} \cong 0.05\, d^4. \tag{7.18}$$

Circular ring. The axial moment of inertia of a circular ring (Fig. 92*b*) is determined as the difference between the moments of inertia of the large circle of diameter D and the small circle of

diameter d

$$I_x = I_y = \frac{\pi D^4}{64} - \frac{\pi d^4}{64} = \frac{\pi}{64}(D^4 - d^4) \cong 0.05\,(D^4 - d^4). \qquad (7.19)$$

In solving practical problems it is often found more convenient to use a formula involving not D and d but D and the ratio $d/D = \alpha$, i. e.,

$$I_x = I_y = \frac{\pi D^4}{64}(1 - \alpha^4) \cong 0.05 D^4 (1 - \alpha^4). \qquad (7.20)$$

If a ring is thin-walled (see Sec. 40), then

$$I_x = I_y = \frac{\pi D^3}{8}\delta \cong 0.4 D^3 \delta. \qquad (7.20')$$

51. Determination of Moments of Inertia of Figures Composed of Simple Figures

In this section we shall consider several examples of determining the moments of inertia of areas composed of simple figures on the basis of the formulas derived in the preceding sections.

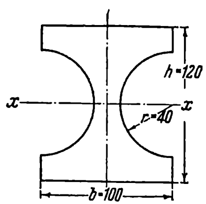


Fig. 93

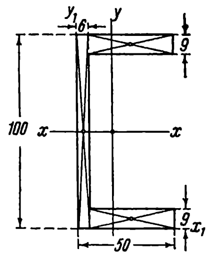


Fig. 94

Example 46. Determine the moment of inertia of the section shown in Fig. 93 with respect to an axis x-x. The dimensions are indicated in millimetres.

Solution. The moment of inertia of the figure is determined as the difference between the moments of inertia of a rectangle bh and a circle of diameter d ($r = 4$ cm).

On the basis of formulas (7.13) and (7.18) we have

$$I_x = \frac{bh^3}{12} - \frac{\pi}{64}d^4 = \frac{10 \times 12^3}{12} - \frac{\pi}{64} \times 8^4 = 1{,}440 - 203 = 1{,}237 \text{ cm}^4.$$

Example 47. Determine the centroidal axial moments of inertia of a channel (Fig. 94). The dimensions are indicated in millimetres.

Solution. Break the section into three rectangles as shown in the drawing. The centroid of the channel is on the axis of symmetry x-x. To determine the distance of the centroid from the y_1 axis we apply formula (7.6), after first determining the static moments of the three rectangles with respect to the y_1 axis and the area of the entire section.

The static moment of the vertical rectangle is

$$S'_{y_1} = 10\times 0.6\times 0.3 = 1.8 \text{ cm}^3.$$

The static moment of one horizontal rectangle is

$$S''_{y_1} = (5-0.6)\,0.9\left(\frac{5-0.6}{2}+0.6\right) = 4.4\times 0.9\times 2.8 = 11.1 \text{ cm}^3.$$

The cross-sectional area of the entire channel is

$$A = 10\times 0.6 + 2\,(5-0.6)\,0.9 = 6 + 7.92 = 13.92 \text{ cm}^2.$$

The distance of the centroid of the channel from the axis is

$$x_c = \frac{S'_{y_1}+2S''_{y_1}}{A} = \frac{1.8+2\times 11.1}{13.92} = 1.72 \text{ cm}.$$

Determine now the centroidal moments of inertia using the three rectangles into which the channel was broken above.

The moments of inertia of the vertical rectangle with respect to the centroidal axes are, respectively,

$$I'_x = \frac{0.6\times 10^3}{12} = 50 \text{ cm}^4,$$

$$I'_y = \frac{10\times 0.6^3}{12} + 10\times 0.6\left(1.72-\frac{0.6}{2}\right)^2 = 0.18 + 12.1 \cong 12.3 \text{ cm}^4.$$

The moments of inertia of the horizontal rectangle are

$$I''_x = \frac{(5-0.6)\,0.9^3}{12} + (5-0.6)\,0.9\left(\frac{10}{2}-\frac{0.9}{2}\right)^2 = 0.267 + 82 \cong 82.3 \text{ cm}^4,$$

$$I''_y = \frac{0.9\,(5-0.6)^3}{12} + (5-0.6)\,0.9\left(\frac{5-0.6}{2}+0.6-1.72\right)^2 =$$
$$= 6.38 + 1.2 = 7.58 \text{ cm}^4.$$

The moments of inertia of the entire channel section with respect to the centroidal axes are

$$I_x = I'_x + 2I''_x = 50 + 2\times 82.3 = 214.6 \text{ cm}^4,$$
$$I_y = I'_y + 2I''_y = 12.3 + 2\times 7.6 = 27.5 \text{ cm}^4.$$

The moment of inertia of the channel section with respect to the axis x-x could have been determined in a simpler way regarding the channel section as the difference between a rectangle of base

5 cm and height 10 cm and a rectangle of base (5—0.6) cm and height (10—2×0.9) cm

$$I_x=\frac{5\times10^3}{12}-\frac{(5-0.6)\,(10-2\times0.9)^3}{12}=416-202=214\ \text{cm}^4.$$

Example 48. Determine the moment of inertia of the cross-sectional area with respect to an axis x-x for a section composed of a vertical web, four No. 5 equal angles of wall thickness 5 mm and two horizontal plates (Fig. 95). The dimensions are indicated in millimetres.

Solution. The moment of inertia of the vertical plate is

$$I'_x=\frac{1\times30^3}{12}=2{,}250\ \text{cm}^4.$$

The moment of inertia of the horizontal plate is

$$I''_x=\frac{13\times1^3}{12}+13\times1\left(15+\frac{1}{2}\right)^2=$$
$$=1.08+3{,}120=3{,}121\ \text{cm}^4.$$

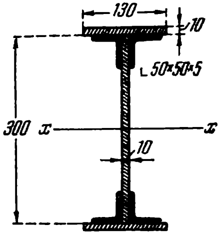


Fig. 95

From the standard sections table for GOST (USSR State Standard) 8509-57 equal angles we find the cross-sectional area, the moment of inertia with respect to a horizontal axis through the centroid of an angle and the distance of the centroid to the base of an angle

$$A=4.8\ \text{cm}^2,\qquad I=11.2\ \text{cm}^4,\qquad y=1.42\ \text{cm}.$$

The moment of inertia of the angle with respect to the axis x-x is

$$I'''_x=I+Aa^2=11.2+4.8\left(\frac{30}{2}-1.42\right)^2=896\ \text{cm}^4.$$

The moment of inertia of the entire section is

$$I_x=I'_x+I''_x+I'''_x=2{,}250+2\times3{,}121+4\times896=12{,}076\ \text{cm}^4.$$

52. Transformation Formulas for Moments of Inertia in the Case of Rotation of Axes

Let the moments of inertia I_x, I_y and I_{xy} for some figure with respect to the co-ordinate axes x and y be known (Fig. 96). It is desired to determine the same moments of inertia with respect to other axes x_1 and y_1 rotated about the x and y axes through an angle α, i.e., I_{x_1}, I_{y_1} and $I_{x_1y_1}$.

Isolate from the section an elementary area dA enclosing a point A of co-ordinates (x, y) with respect to the old co-ordinate system

$$x = \overline{OB}, \quad y = \overline{AB}.$$

The co-ordinates of the same area with respect to the new co-ordinate system are

$$x_1 = \overline{OC}, \quad y_1 = \overline{AC}.$$

We express the new co-ordinates x_1 and y_1 in terms of the old co-ordinates x and y and the angle of rotation α. Drawing auxiliary

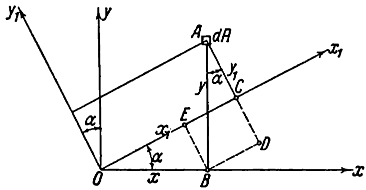


Fig. 96

lines $\overline{CD}$ and $\overline{BE}$ parallel to the axis Oy_1 and $\overline{BD}$ parallel to the axis Ox_1, we obtain

$$\overline{OC} = \overline{OE} + \overline{EC} = \overline{OE} + \overline{BD},$$
$$\overline{AC} = \overline{AD} - \overline{CD} = \overline{AD} - \overline{BE}.$$

Substituting the values of the quantities involved

$$\overline{OC} = x_1, \quad \overline{OE} = \overline{OB}\cos\alpha = x\cos\alpha,$$
$$\overline{BD} = \overline{AB}\sin\alpha = y\sin\alpha, \quad \overline{AC} = y_1,$$
$$\overline{AD} = \overline{AB}\cos\alpha = y\cos\alpha, \quad \overline{BE} = \overline{OB}\sin\alpha = x\sin\alpha,$$

we obtain

$$x_1 = x\cos\alpha + y\sin\alpha,$$
$$y_1 = y\cos\alpha - x\sin\alpha.$$

By definition, the desired moments of inertia with respect to the new axes are

$$I_{x_1} = \int_A y_1^2 dA, \quad I_{y_1} = \int_A x_1^2 dA, \quad I_{x_1y_1} = \int_A x_1 y_1 dA.$$

Substituting the values of x_1 and y_1 gives

$$I_{x_1}=\int_A (y\cos\alpha-x\sin\alpha)^2\,dA=$$
$$=\cos^2\alpha\int_A y^2dA+\sin^2\alpha\int_A x^2dA-2\sin\alpha\cos\alpha\int_A xydA=$$
$$=I_x\cos^2\alpha+I_y\sin^2\alpha-I_{xy}\sin 2\alpha. \tag{7.21}$$

In a similar way we obtain an expression for the moment of inertia with respect to the other axis

$$I_{y_1}=\int_A (x\cos\alpha+y\sin\alpha)^2\,dA=$$
$$=\cos^2\alpha\int_A x^2dA+\sin^2\alpha\int_A y^2dA+2\sin\alpha\cos\alpha\int_A xydA=$$
$$=I_y\cos^2\alpha+I_x\sin^2\alpha+I_{xy}\sin 2\alpha. \tag{7.22}$$

Adding and subtracting expressions (7.21) and (7.22), we find

$$I_{x_1}+I_{y_1}=I_x(\sin^2\alpha+\cos^2\alpha)+I_y(\sin^2\alpha+\cos^2\alpha),$$
$$I_{x_1}-I_{y_1}=I_x(\cos^2\alpha-\sin^2\alpha)-I_y(\cos^2\alpha-\sin^2\alpha)-2I_{xy}\sin 2\alpha;$$

since

$$\sin^2\alpha+\cos^2\alpha=1,$$
$$\cos^2\alpha-\sin^2\alpha=\cos 2\alpha,$$

we have

$$I_{x_1}+I_{y_1}=I_x+I_y, \tag{7.23}$$
$$I_{x_1}-I_{y_1}=(I_x-I_y)\cos 2\alpha-2I_{xy}\sin 2\alpha. \tag{7.24}$$

Equality (7.23) expresses the property of the sum of the moments of inertia with respect to two perpendicular axes which was previously derived in Sec. 48 in a different way. Formulas (7.23) and (7.24) may conveniently be used to determine the moments of inertia I_{x_1} and I_{y_1}.

Determine now the magnitude of the product of inertia in exactly the same way as was done for the axial moments of inertia

$$I_{x_1y_1}=\int_A x_1y_1dA=\int_A (x\cos\alpha+y\sin\alpha)(y\cos\alpha-x\sin\alpha)\,dA=$$
$$=\cos^2\alpha\int_A xydA+\sin\alpha\cos\alpha\int_A y^2dA-$$
$$-\sin\alpha\cos\alpha\int_A x^2dA-\sin^2\alpha\int_A xydA=$$
$$=\sin\alpha\cos\alpha\left(\int_A y^2dA-\int_A x^2dA\right)+(\cos^2\alpha-\sin^2\alpha)\int_A xydA=$$
$$=\frac{I_x-I_y}{2}\sin 2\alpha+I_{xy}\cos 2\alpha. \tag{7.25}$$

Let us group together the transformation formulas for moments of inertia in the case of rotation of axes through an angle α

$$I_{x_1}=I_x\cos^2\alpha+I_y\sin^2\alpha-I_{xy}\sin 2\alpha, \tag{7.21}$$

$$I_{y_1}=I_y\cos^2\alpha+I_x\sin^2\alpha+I_{xy}\sin 2\alpha, \tag{7.22}$$

$$I_{x_1y_1}=\frac{I_x-I_y}{2}\sin 2\alpha+I_{xy}\cos 2\alpha. \tag{7.25}$$

These formulas are fundamental ones in design problems involving the determination of moments of inertia with respect to any axes.

53. Concept of Principal Axes of Inertia and Determination of Their Position

From formulas (7.21), (7.22) and (7.25) it is seen that moments of inertia I_{x_1}, I_{y_1} and $I_{x_1y_1}$ depend on the angle α. As the angle of rotation of the axes α varies, so do the magnitudes of moments of inertia.

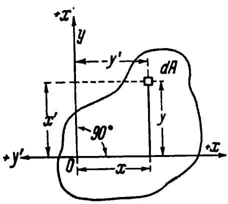


Fig. 97

We shall first show that when the co-ordinate axes are rotated through 90° the sign of the product of inertia is reversed. Indeed, suppose, for example, that the co-ordinate axes x and y for some figure (Fig. 97) are rotated counter-clockwise about the origin through 90° and occupy positions x' and y'. The new co-ordinates of an elementary area dA are expressed in terms of the old ones as follows:

$$x'=+y,$$

$$y'=-x.$$

The product of inertia with respect to the new axes x' and y' is

$$I_{x_1y_1}=\int\limits_A x'y'\,dA=-\int\limits_A xy\,dA=-I_{xy}.$$

This could have been proved by substituting an angle $(\alpha+90°)$ for α in formula (7.25).

The magnitude of the product of inertia varies continuously with the angle of rotation of the co-ordinate axes. When the axes are rotated through 90° the magnitude of the product of inertia changes sign; consequently, when passing from one sign to its opposite there must be a position of axes for which it is equal to zero.

Axes with respect to which the product of inertia is zero are called ***principal axes of inertia***. If the origin coincides with the

centroid of a figure, the corresponding principal axes are called *principal centroidal axes of inertia.*

If a figure has an axis of symmetry, this axis is always one of the principal axes. Indeed, let the y axis be an axis of symmetry of the section shown in Fig. 98. Take an elementary area dA of

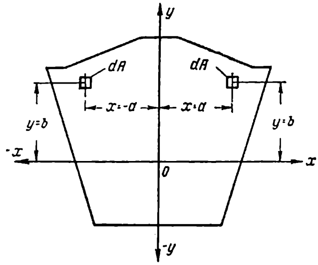


Fig. 98

co-ordinates $x=a$, $y=b$. The product of inertia of this elementary area is

$$dI_{xy}=ab\,dA.$$

On the other side of the axis there is always an elementary area dA having the co-ordinates $x=-a$, $y=b$; its product of inertia is

$$dI'_{xy}=-ab\,dA.$$

The sum of the products of inertia of these two symmetrically located areas is zero

$$dI_{xy}+dI'_{xy}=0.$$

This is true for each pair of symmetrically located elementary areas. Therefore, the product of inertia of the entire figure composed of symmetrical elementary areas is zero. If, however, the product of inertia is zero, the axis of symmetry and any axis perpendicular to it are principal axes.

Thus, the location of principal axes of inertia for symmetrical figures presents no difficulty.

Consider now the problem of determining the position of principal axes of inertia in the general case when a figure has no axis of symmetry.

Suppose that the moments of inertia I_x, I_y and I_{xy} with respect to arbitrary axes x and y are known. The product of inertia with

respect to other co-ordinate axes x_1 and y_1 having the same origin but rotated with respect to the former through an angle α is, according to formula (7.25),

$$I_{x_1y_1}=\frac{I_x-I_y}{2}\sin 2\alpha+I_{xy}\cos 2\alpha.$$

Equating the product of inertia $I_{x_1y_1}$ to zero, we find an angle through which the original axes must be rotated to make the new axes principal

$$\frac{I_x-I_y}{2}\sin 2\alpha+I_{xy}\cos 2\alpha=0,$$

whence

$$\tan 2\alpha=\frac{2I_{xy}}{I_y-I_x}. \tag{7.26}$$

Substituting the values of I_x, I_y and I_{xy} in this formula, we find two values for the angle 2α differing by 180°. The angles α themselves will differ by 90°. Consequently, the principal axes will be perpendicular to each other.

Thus, to determine the position of the principal axes of inertia it is necessary, in general, to know the moments of inertia I_x, I_y and I_{xy} with respect to any pair of co-ordinate axes.

Let us now show that the axial moments of inertia with respect to principal axes of inertia assume limiting values, either the maximum or the minimum value. For this purpose we find an angle of rotation α for which the moments of inertia are maximum or minimum. Take expression (7.21) for the axial moment of inertia with respect to an axis rotated through an angle α relative to the original position

$$I_{x_1}=I_x\cos^2\alpha+I_y\sin^2\alpha-I_{xy}\sin 2\alpha.$$

Find the derivative $dI_{x_1}/d\alpha$ and equate it to zero

$$\frac{dI_{x_1}}{d\alpha}=-2I_x\sin\alpha\cos\alpha+2I_y\sin\alpha\cos\alpha-2I_{xy}\cos 2\alpha=0$$

or

$$(I_y-I_x)\sin 2\alpha=2I_{xy}\cos 2\alpha.$$

Hence

$$\tan 2\alpha=\frac{2I_{xy}}{I_y-I_x}.$$

The expression obtained for the angle of rotation is the same as that for the principal axes [formula (7.26)]. Consequently, the proposition stated above is proved. If the moment of inertia with respect to a new axis rotated through an angle α defined by the expression obtained has a maximum value, the moment of inertia

with respect to the other, perpendicular axis has a minimum value, and vice versa. This follows from the fact that the sum of the axial moments with respect to two perpendicular axes remains unchanged when these axes are rotated about the origin [formulas (7.8) and (7.23)].

We can now give an alternative definition of principal axes: *principal axes of inertia are two perpendicular axes with respect to which the axial moments of inertia have a maximum and a minimum value.*

Sometimes formula (7.26) is replaced by a formula which follows from it

$$\tan \alpha_1 = \frac{I_{xy}}{I_{\min}-I_x}; \tag{7.27}$$

this formula enables one to find an angle α_1 through which the x axis must be rotated to obtain an axis giving $I_{\max}$.

54. Determination of Principal Moments of Inertia

If the moments of inertia I_x, I_y and I_{xy} of a figure with respect to any co-ordinate axes are known, the magnitudes of the principal moments of inertia can be determined as follows.

Determine first the positions of the principal axes of inertia by the formula

$$\tan 2\alpha = \frac{2I_{xy}}{I_y-I_x}. \tag{7.26}$$

After determining the angle 2α find the values of $\sin 2\alpha$ and $\cos 2\alpha$ from tables.

From the preceding discussion it is known that the sum and the difference of the moments of inertia with respect to the rotated axes are given by formulas (7.23) and (7.24)

$$I_{x_1}+I_{y_1}=I_x+I_y, \tag{7.23}$$

$$I_{x_1}-I_{y_1}=(I_x-I_y)\cos 2\alpha-2I_{xy}\sin 2\alpha. \tag{7.24}$$

Substituting the values of $\sin 2\alpha$ and $\cos 2\alpha$ in formula (7.24), we obtain a system of two equations from which the principal moments of inertia I_{x_1} and I_{y_1} are easily determined.

It is possible, however, to carry out the solution in the general form and obtain the following formulas for the principal moments of inertia:

$$I_{\max}=\frac{I_x+I_y}{2}+\sqrt{\left(\frac{I_x-I_y}{2}\right)^2+I_{xy}^2}, \tag{7.28}$$

$$I_{\min}=\frac{I_x+I_y}{2}-\sqrt{\left(\frac{I_x-I_y}{2}\right)^2+I_{xy}^2}. \tag{7.29}$$

These formulas are obtained from equalities (7.21) and (7.26) by eliminating the angle α.

If the moments of inertia with respect to the principal axes x_1 and y_1 are found, the moments of inertia with respect to any other axes x, y rotated through an angle α are determined from formulas (7.21), (7.22) and (7.25)

$$I_x = I_{x_1}\cos^2\alpha + I_{y_1}\sin^2\alpha, \tag{7.30}$$

$$I_y = I_{y_1}\cos^2\alpha + I_{x_1}\sin^2\alpha, \tag{7.31}$$

$$I_{xy} = \frac{I_{x_1} - I_{y_1}}{2}\sin 2\alpha. \tag{7.32}$$

These formulas differ from formulas (7.21), (7.22) and (7.25) in that they involve no term containing the product of inertia which is zero for the principal axes x_1, y_1.

It will readily be seen that the equations of the theory of moments of inertia are similar in form to those of the theory of combined stresses treated in Chap. IV. Thus, Eqs. (4.11) and (4.12a), which define the normal and shearing stresses on an inclined plane, are similar to Eqs. (7.21) and (7.25), which define the moments of inertia for rotated axes. This is also true of equations for determining the positions of the principal planes and principal axes [Eqs. (4.13) and (7.26)] or of equations for the principal stresses (4.14) and the principal moments of inertia (7.28), (7.29). This analogy is extended to the properties considered: thus, the sum of the equatorial moments of inertia for perpendicular axes passing through a given origin is constant, and so is the sum of the normal stresses on two perpendicular planes passed through a given point.

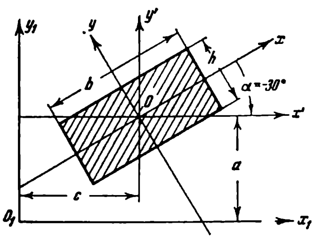


Fig. 99

Example 49. Determine the axial moments of inertia of a rectangle of sides $b = 9$ cm, $h = 4$ cm with respect to x_1 and y_1 axes if $\alpha = 30°$, $a = 10$ cm and $c = 8$ cm (Fig. 99).

Solution. The moments of inertia of the rectangle with respect to the principal centroidal x and y axes are

$$I_x = \frac{bh^3}{12} = \frac{9\times 4^3}{12} = 48 \text{ cm}^4; \quad I_y = \frac{hb^3}{12} = \frac{4\times 9^3}{12} = 243 \text{ cm}^4.$$

The moments of inertia with respect to the rotated axes x' and y' are determined on the basis of formulas (7.30) and (7.31). Since these formulas contain $\sin\alpha$ and $\cos\alpha$ to the second power, the direction of rotation of the axes is unimportant and we obtain

$$I_{x'} = I_x \cos^2 30^\circ + I_y \sin^2 30^\circ =$$
$$= 48\times 0.866^2 + 243\times 0.5^2 = 36 + 60.7 = 96.7 \text{ cm}^4,$$
$$I_{y'} = I_y \cos^2 30^\circ + I_x \sin^2 30^\circ =$$
$$= 243\times 0.866^2 + 48\times 0.5^2 = 182 + 12 = 194 \text{ cm}^4.$$

The moments of inertia of the rectangle with respect to the x_1 and y_1 axes are, according to formula (7.10),

$$I_{x_1} = I_{x'} + Aa^2 = 96.7 + 9\times 4\times 10^2 = 3{,}700 \text{ cm}^4,$$
$$I_{y_1} = I_{y'} + Ac^2 = 194 + 9\times 4\times 8^2 = 2{,}490 \text{ cm}^4.$$

Example 50. Determine the positions of the principal centroidal axes of inertia and the magnitudes of the principal moments of inertia of a Z-section (Fig. 100). All dimensions of the section are indicated in millimetres.

Solution. Determine first the moments of inertia I_x, I_y and I_{xy} with respect to the x and y axes. To do this we break the section into a vertical rectangle and two horizontal rectangles as shown in Fig. 100.

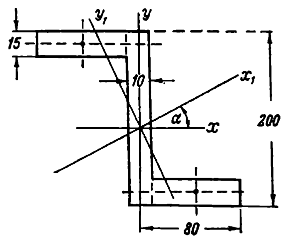


Fig. 100

The moments of inertia of the vertical rectangle with respect to the x and y axes are

$$I'_x = \frac{1\times 20^3}{12} = 667 \text{ cm}^4,$$

$$I'_y = \frac{20\times 1^3}{12} = 1.66 \text{ cm}^4.$$

The moments of inertia of one horizontal rectangle are

$$I''_x = \frac{(8-1)\,1.5^3}{12} + (8-1)\,1.5\left(\frac{20}{2}-\frac{1.5}{2}\right)^2 = 1.97 + 898 \cong 900 \text{ cm}^4,$$
$$I''_y = \frac{1.5\,(8-1)^3}{12} + (8-1)\,1.5\left(\frac{8-1}{2}+\frac{1}{2}\right)^2 = 30.4 + 168 \cong 198 \text{ cm}^4.$$

The moments of inertia of the entire section are

$$I_x = I'_x + 2I''_x = 667 + 2\times 900 \cong 2{,}467 \text{ cm}^4,$$
$$I_y = I'_y + 2I''_y = 1.66 + 2\times 198 \cong 398 \text{ cm}^4.$$

The product of inertia of the vertical rectangle with respect to the x and y axes is zero since these axes are principal axes for it.

Determine the products of inertia of the horizontal rectangles. The products of inertia with respect to their axes of symmetry shown dashed in the drawing are zero. In transferring to the x and y axes we obtain on the basis of formula (7.11):

for the top horizontal rectangle

$$I'_{xy} = (8-1)\,1.5\left[-\left(\frac{8-1}{2}+\frac{1}{2}\right)\right]9.25 =$$
$$= 7\times 1.5\,(-4)\,9.25 = -388 \text{ cm}^4,$$

for the bottom horizontal rectangle

$$I''_{xy} = (8-1)\,1.5\left(\frac{8-1}{2}+\frac{1}{2}\right)\left[-\left(\frac{20}{2}-\frac{1.5}{2}\right)\right] =$$
$$= 7\times 1.5\times 4\,(-9.25) = -388 \text{ cm}^4.$$

The product of inertia of the entire section is

$$I_{xy} = I'_{xy} + I''_{xy} = -388 + (-388) = -776 \text{ cm}^4.$$

The positions of the principal centroidal axes of inertia are determined by formula (7.26)

$$\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x} = \frac{2(-776)}{398-2{,}467} = 0.75.$$

From a table of trigonometric functions we find

$$2\alpha = 36°52', \qquad \alpha = 18°26',$$
$$2\alpha = 216°52', \qquad \alpha = 108°26'.$$

From the same table we write out the values

$$\sin 2\alpha = \pm 0.6, \qquad \cos 2\alpha = \pm 0.8.$$

On the basis of formulas (7.23) and (7.24) we have

$$I_{x_1} + I_{y_1} = I_x + I_y = 398 + 2{,}467 = 2{,}865 \text{ cm}^4,$$
$$I_{x_1} - I_{y_1} = (I_x - I_y)\cos 2\alpha - 2I_{xy}\sin 2\alpha =$$
$$= (2{,}467 - 398)(\pm 0.8) - 2(-770)(\pm 0.6)$$

or

$$I_{x_1} + I_{y_1} = 2{,}865 \text{ cm}^4,$$
$$I_{x_1} - I_{y_1} = \pm 2{,}585 \text{ cm}^4.$$

Adding and subtracting these equations, with the plus sign taken on the right-hand side of the second equation, we obtain

$$I_{x_1} = 2{,}725\ \text{cm}^4, \qquad I_{y_1} = 140\ \text{cm}^4.$$

The moment of inertia I_{x_1} is maximum and I_{y_1} minimum.

If the minus sign is taken on the right-hand side of the second equation, we obtain

$$I_{x_1} = 140\ \text{cm}^4, \qquad I_{y_1} = 2{,}725\ \text{cm}^4,$$

but the x_1 and y_1 axes will be rotated through 90° with respect to the x_1 and y_1 axes shown in Fig. 100.

55. Check Questions

How is the static moment of a figure determined in terms of the area of the figure and the co-ordinates of its centroid? What is the dimension of the static moment?

What is the static moment of a figure with respect to an axis through the centroid of the figure?

What formulas are used to determine the co-ordinates of the centroid of a figure?

What are the axial and polar moments of inertia and the product of inertia? What are their dimensions?

What is the relationship between the sum of the axial moments of inertia with respect to perpendicular axes and the polar moment of inertia with respect to the point of intersection of these axes?

What moments of inertia are always positive?

Write the transformation formulas for the axial moment of inertia and the product of inertia in the case of parallel transfer of axes.

What is the axial moment of inertia of a rectangle with respect to its centroidal axis parallel to the base?

What are the centroidal axial moments of inertia of a circle and a circular ring?

How is the product of inertia changed when the co-ordinate axes are rotated through 90°?

What axes are called principal centroidal axes of inertia?

Why is an axis of symmetry of a figure always one of the principal axes of inertia?

How is the moment of inertia of a composite figure determined if it can be broken into simple figures whose moments of inertia are easily determined from formulas or tables?

Chapter VIII

Bending of a straight rod, bending moment and shearing force

56. General Considerations

Consider a straight prismatic rod with a longitudinal plane of symmetry (Fig. 101); apply in this plane balanced forces acting perpendicular to the rod axis. Under the action of these forces the rod bends, its axis deflects. Such bending of the rod is called

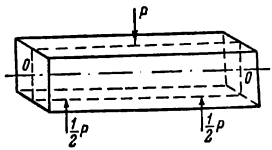


Fig. 101

transverse bending. The bending deformation of the rod takes place in the plane of action of the forces; hence the bending is called plane bending.

As an example consider, under these conditions, a rod bent by four equal forces P acting perpendicular to the rod axis and lying in the same plane (Fig. 102*a*). It is readily seen that the rod is in equilibrium under the action of the forces applied to it. Imagine the rod cut through a section mn located in portion BC.

Consider now either of the parts of the rod, say, the left one (Fig. 102*b*); this part of the rod is acted on by a couple of moment Pa which tends to rotate it clockwise. In order for this part of the rod to maintain the state of equilibrium in which it was before cutting, it is necessary to apply at section mn a moment M of elastic forces equal in magnitude to the moment Pa of the external forces but acting in the opposite sense. Wherever in portion BC we pass a section through the rod, we get the same answer, namely, the cut-off part of the rod will always be in equilibrium under the action of two moments equal in magnitude to Pa and opposite in sense. The bending of a rod (portion BC

in this case) produced by two equal moments of opposite sense is called *pure bending*.

Consequently, the portion BC of the rod is in a state of pure bending.

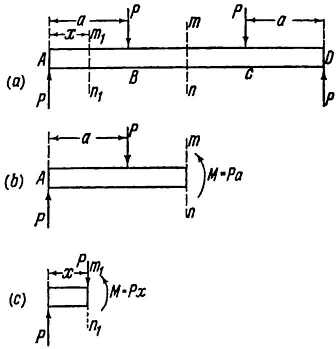


Fig. 102

The situation is different if we pass a section in a portion AB or CD of the rod. Indeed, cut the rod in portion AB through a section m_1n_1 at a distance x from the left end (Fig. 102*c*). The left part of the rod will be in equilibrium if we apply at section m_1n_1 a moment M of elastic forces equal to the moment Px of the external force but acting in the opposite sense and a resultant of elastic forces directed downward and equal to P. The force P acting in the plane of the section tends to shear off the rod along this section. Consequently, in this case bending produced by the moments is accompanied by shearing deformation. This type of deformation is called *transverse bending*.

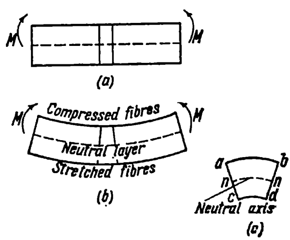


Fig. 103

To visualize the bending deformation let us consider a small prismatic rubber bar. We mark on its face two lines parallel to each other and perpendicular to the axis of the bar. Apply two

equal and opposite moments at its ends in the plane of symmetry (Fig. 103*a*). The bar is deflected under the action of the bending moments, the straight lines marked on the surface remain straight and perpendicular to the deflected axis of the bar (Fig. 103*b*).

This simple experiment provides a great deal of information. It may be concluded, assuming the phenomenon occurring in the interior of the beam in the same way as on its faces, that *plane cross sections remain plane during bending*. Further it is seen that these *plane sections rotate with respect to one another*. Obviously, this rotation is due to extension of some fibres of the material and compression of others. In our example the top fibres (on the concave side of the bar) are compressed. From this it is readily concluded that there is a layer of fibres in the beam which undergoes neither extension nor compression. This layer is referred to as the *neutral layer*. The line of intersection of the neutral layer and the plane of any cross section is called the neutral axis. In Fig. 103*c* line *nn* is the neutral axis.

Furthermore, from the same rubber model it is easily seen that the longitudinal contraction on fibres on the concave side is accompanied by elongation in the transverse direction, and the longitudinal elongation of fibres on the convex side by contraction in the transverse direction, i. e., the phenomena occur in the same way as in the case of simple tension and compression. As a result, the top and bottom sides of the section, i. e., lines *ab* and *cd* become curved: the top line *ab* is lengthened and the bottom line *cd* is shortened.

These very valuable and, it would seem, simple conclusions were not drawn by scientists all at once. It took more than a century from the beginning of the study of bending to arrive at a correct understanding of the bending phenomenon. Galileo Galilei, who first began to study the bending theory back in the XVII century, made the faulty assumption that all fibres of material were lengthened in the same way during bending. It was not until the end of the XVIII century that the correct assumption made at the beginning of the same century was confirmed experimentally, namely, that some fibres, on the convex side, are stretched, and the others, on the concave side, are compressed during bending.

Due to the elongation of some fibres in a rod and the contraction of others caused by bending moments, tensile and compressive normal stresses occur on cross sections of the rod. The magnitude of these stresses at any cross section depends on the magnitude of the bending moment acting at this section. We saw above that, in the case of bending of a rod by forces, in addition to bending moments there act shearing forces at cross sections which tend to produce shearing deformation. Shearing forces give rise to shearing stresses on cross sections of the rod whose magnitude depends on

the magnitude of the shearing force at a given section. In general, then, both normal and shearing stresses will occur in a rod bent by forces.

Before proceeding to the determination of these stresses let us consider some methods of determining bending moments and shearing forces at different cross sections of bent rods.

57. Supports and Reactions at Supports of Beams

Straight rods resting on supports and bent by loads applied to them are usually called beams. Beams serve to transmit loads acting on them to supports on which they are resting. The supports of a beam exert reactions which must be determined first in solving all problems dealing with the bending of beams. The number of reactions to be determined depends on the number and construction of supports of a beam.

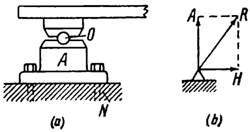


Fig. 104

Supports of beams may be divided into the following three basic types according to their construction:

(1) immovable hinged support;
(2) movable hinged support;
(3) fixed support.

An immovable hinged support is shown in Fig. 104*a*. The end of the beam is supported by a hinge O. The latter rests on a bearing pad A, which in turn is rigidly fastened to foundation N. This support permits no movement of the end of the beam in any direction except for rotation about the centre of the hinge O. Henceforth, an immovable hinged support will be represented schematically as indicated in Fig. 104*b*.

As to the reaction arising at an immovable hinged support we only know that it lies in the plane of action of forces applied to the beam and passes through the centre of the hinge. The magnitude and direction of the reaction are not known. The reaction R, which is unknown in magnitude and direction, can always be replaced by two component reactions, one being a vertical reaction A, and the other a horizontal reaction H. In this case, instead of the reaction unknown in magnitude and direction, we obtain two reactions known in direction but unknown in magnitude. Thus, it may be said that an immovable hinged support involves two reactions of unknown magnitude. A movable hinged support is shown in Fig. 105*a*. This support differs from an immovable hinged support in that its bearing pad is placed on rollers which permit it to move together with the end of the beam along the axis

of the latter on the supporting plane N. Henceforth, a movable hinged support will be represented schematically as indicated in Fig. 105*b*. A movable hinged support imposes only one constraint on the end of the beam, namely, it permits no movement of the end of the beam in a direction perpendicular to the axis of the

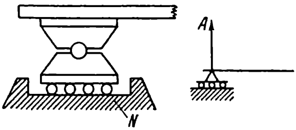


Fig. 105

beam. Consequently, a movable hinged support involves only one reaction unknown in magnitude but known in direction.

A fixed support of the end of a beam is shown schematically in Fig. 106. This support prevents any movement of the end of the beam in the plane of action of external loads and, also, prevents rotation of the end of the beam. A fixed support produces a reaction unknown in magnitude and direction which prevents any movement of the end of the beam and a reactive moment which prevents rotation of the end of the beam. The unknown reaction R can always be replaced by two reactions, one being a vertical reaction A, and the other a horizontal reaction H. On this basis it may be said that a fixed support involves three unknown reactions: vertical reaction A, horizontal reaction H and support moment m.

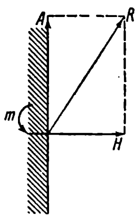


Fig. 106

In practice bending is commonly produced by forces perpendicular to the axis of a beam. In these cases the number of unknown reactions arising at supports is reduced since the reactions along the axis of the beam at an immovable hinged support and at a fixed support become equal to zero. Thus, for beams subjected to bending loads perpendicular to the axis of the beam we have one unknown reaction A at an immovable or a movable hinged support, which is perpendicular to the axis of the beam, and two unknown reactions at a fixed support, a reaction A perpendicular to the axis of the beam and a reactive moment m.

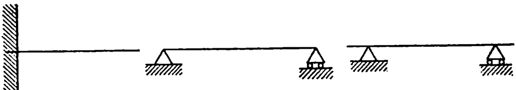
58. Determination of Reactions at Supports of Beams

Since the strains under all types of deformation studied in strength of materials are assumed to be small, the changes produced by the deformation in the position of external forces acting on a beam may be neglected in the determination of reactions at supports.

In the case of a beam subjected to forces lying in the same plane statics gives three equations of equilibrium

$$\Sigma X = 0, \qquad \Sigma Y = 0, \qquad \Sigma M = 0,$$

i. e., for equilibrium of the beam the sums of projections of the x and y axes of all forces applied to the beam, including the re-



(a) (b) (c)

Fig. 107

actions at supports, must be zero; besides, the sum of moments of all forces about any point in the plane must also be zero.

If the forces acting on the beam are perpendicular to its axis, the equation $\Sigma X = 0$ becomes an identity and only two equations of statics are left for determining reactions

$$\Sigma Y = 0, \qquad \Sigma M = 0. \tag{8.1}$$

If a beam subjected to transverse bending has such supports that the total number of reactions arising at them does not exceed two, the reactions can always be determined from two equations (8.1) of statics. Beams whose reactions can be determined from equations of statics are called *statically determinate beams*. Statically determinate beams may be of the following two types: (1) a beam with one end fixed and the other free, or a *cantilever* (Fig. 107*a*), and (2) a beam with an immovable hinged support at one end and a movable hinged support at the other (Fig. 107*b* and *c*).

The beam shown in Fig. 107*c* has overhanging ends. Such a beam is called a *cantilever beam* and the overhanging ends *cantilevers*. The beam of Fig. 107*b* is termed a *simple* beam.

Beams in which the total number of reactions at supports is greater than the number of equations of static equilibrium are called *statically indeterminate beams*. In the case of statically indeterminate beams the reactions at supports are determined by sol-

ving simultaneously the equations of statics and the deformation equations for beams. Therefore, the determination of reactions of statically indeterminate beams will be taken up later, after we learn to determine deformations of beams. The procedures for determining reactions of statically determinate beams will now be illustrated by examples.

We first agree to choose the x axis always along the axis of a beam and the y axis vertically upward (Fig. 108). In setting up moment equations clockwise moments are considered as positive. If a uniformly distributed continuous load acts on a beam, as shown in Fig. 108, the continuous load is replaced by its resultant in the determination of reactions. An example of a uniformly distributed continuous load is the weight of a beam. The point of application of a uniformly distributed continuous load is at the centre of the portion on which it is acting. A uniformly distributed continuous load is often specified by its intensity.

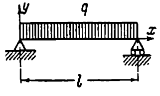


Fig. 108

The *intensity of continuous load* is defined as the load per unit length. If the total continuous load is P and the length of the portion on which it is acting is l, the load intensity is

$$q=\frac{P}{l}.$$

The load intensity q is usually expressed in tons/m, kgf/m or kgf/cm.

When the intensity q of uniformly distributed continuous load and the length of the portion on which it is acting are given, the magnitude of its resultant is determined as the product of the load intensity and the length of the portion

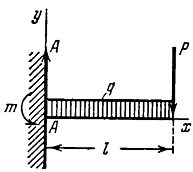


Fig. 109

$$P=ql.$$

Example 51. A beam fixed at one end (Fig. 109) carries a uniformly distributed load of intensity $q=0.5$ ton/m over its entire length and a concentrated force $P=2$ tons at its free end. Determine the reaction at the fixed support if the length of the beam is $l=4$ m.

Solution. The fixed support produces a vertical reaction and a reactive moment. The directions of these reactions are not known. We arbitrarily assume that the vertical reaction A acts upward and the support moment m counter-clockwise. We write equilibrium

conditions, choosing point A as moment centre

$$\Sigma M_A = -m + ql\frac{l}{2} + Pl = 0,$$

whence the magnitude of the reactive moment is

$$m = \frac{ql^2}{2} + Pl = \frac{0.5\times 4^2}{2} + 2\times 4 = 12 \text{ tons-m}.$$

From the force equation of equilibrium in the y direction we obtain

$$A - ql - P = 0,$$

whence the reaction is

$$A = ql + P = 0.5\times 4 + 2 = 4 \text{ tons}.$$

In this case the moment m and the reaction A are positive. This indicates that their directions were chosen correctly. If any one of the reactions thus determined comes out negative, this indicates that the direction arbitrarily assumed for it does not

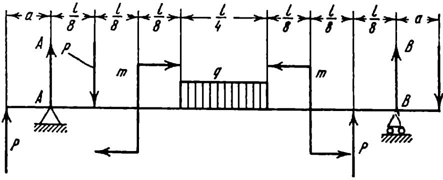


Fig. 110

coincide with the true one. Therefore, the direction of the reaction which is obtained with a minus sign should be reversed in the drawing and further calculations should be based on its true direction.

Example 52. Determine the reactions of the beam shown in Fig. 110.

Solution. The reactions A and B will be arbitrarily directed upward. Write the moment equation about point A

$$\Sigma M_A = 0.$$

From this we find the magnitude of the reaction B

$$B = P\left(\frac{a}{l} + \frac{1}{8} - \frac{7}{8} + \frac{a}{l} + 1\right) + q\frac{l}{8} = P\frac{8a+l}{4l} + q\frac{l}{8}.$$

Write the moment equation about point B

$$\Sigma M_B = 0.$$

From this we find the magnitude of the reaction A

$$A = P\left(\frac{7}{8} - \frac{a}{l} - 1 - \frac{1}{8} - \frac{a}{l}\right) + \frac{ql}{8} = -P\frac{8a+l}{4l} + q\frac{l}{8}.$$

59. Shearing Force and Bending Moment

Consider a beam simply supported at its ends and subjected to two forces, P_1 and P_2 (Fig. 111a). Let the reactions at the left and right supports be A and B, respectively. To determine the internal elastic forces at any section of the beam we apply the general procedure, namely, the method of sections.

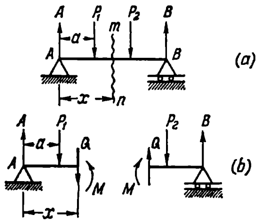


Fig. 111

Imagine the beam to be cut through a section mn at a distance x from its left end and consider the left-hand part of the beam, removing its right-hand part.

In order for the left-hand part of the beam to be in equilibrium, a shearing force Q and a bending moment M must act at the section.

From the equilibrium conditions for the left-hand part of the beam we have

$$(1)\ \Sigma Y = 0, \qquad A - P_1 - Q = 0,$$

whence

$$Q = A - P_1;$$

$$(2)\ \Sigma M = 0, \qquad Ax - P_1(x-a) - M = 0,$$

whence

$$M = Ax - P_1(x-a).$$

The resultant Q of the internal forces which is applied to the remaining part of the beam and numerically equal to the algebraic sum of the external forces acting to one side of the section is called the *transverse* or *shearing force at the section*.

The moment M of the internal forces which is applied to the remaining part of the beam and numerically equal to the algebraic sum of the moments of the external forces acting to one side of the section is called the *bending moment at the section*.

Since the whole beam is in equilibrium under the action of the external forces, including the reactions, the sum of all forces

acting on the part of the beam to the left of the section must be equal and opposite to the sum of the forces acting on the part of the beam to the right of the section.

From the same equilibrium condition, the sum of the moments about the centroid of the section of all forces acting to the left

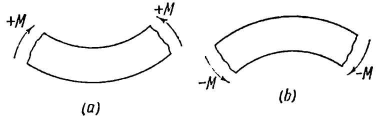


Fig. 112

of the section must be equal and opposite to the sum of the moments of the forces acting to the right of the section about the centroid of the section.

If we consider the right-hand part of the beam instead of the left-hand part, the bending moment and the shearing force at section *mn* will be the same as for the left-hand part but of

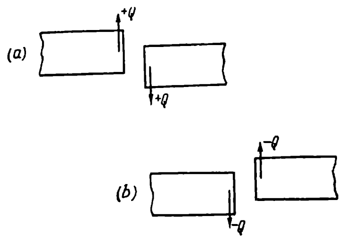


Fig. 113

opposite sense (Fig. 111*c*). Thus, the elastic forces at one and the same section of a beam will have opposite senses depending on the part (right- or left-hand) to which they are applied.

In order to have the same sign for the bending moment or shearing force at one and the same section regardless of the part of a beam to which they are applied, we adopt the following sign rule.

The bending moment is considered positive if it tends to bend a beam convex downward, i. e., if it tends to rotate the left section of the beam clockwise or the right section of the beam counterclockwise (Fig. 112*a*).

The bending moment is considered negative if it tends to bend a beam convex upward, i. e., if it tends to rotate the left section of the beam counter-clockwise or the right section of the beam clockwise (Fig. 112*b*).

The shearing force is considered positive if it tends to move the left section of the beam upward or the right section downward (Fig. 113*a*) as if rotating the two parts of the beam clockwise.

The shearing force is considered negative if it tends to move the left section of the beam downward or the right section upward (Fig. 113*b*).

60. Relations Between Load Intensity, Shearing Force and Bending Moment

Consider a beam simply supported at its ends and subjected to forces P_1, P_2, P_3, P_4 (Fig. 114). Let the reactions at the supports be A and B. Write the moment at a section mn due to the forces lying to the left of the section

$$M_x = Ax - P_1(x - a_1) + P_2(x - a_2).$$

The moments of the forces A and P_2 are positive since they tend to rotate the left-hand part of the beam clockwise; the moment of the force P_1 is negative since it tends to rotate the left-hand part of the beam counter-clockwise.

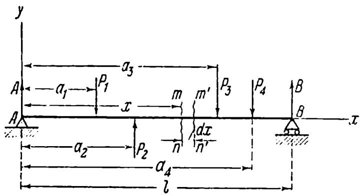


Fig. 114

The shearing force at section mn, being the algebraic sum of all forces lying to the left of the section, is

$$Q = A - P_1 + P_2.$$

Write the value for the moment at a section $m'n'$ a distance dx from section mn

$$M_{x+dx} = A(x + dx) - P_1(x + dx - a_1) + P_2(x + dx - a_2).$$

The increment of the moment is

$$dM = M_{x+dx} - M_x = Adx - P_1dx + P_2dx = (A - P_1 + P_2)\,dx.$$

The quantity in the parentheses represents the shearing force at section mn or at section $m'n'$ since no external force acts between these sections

$$dM = Q\,dx$$

or

$$Q = \frac{dM}{dx}, \tag{8.2}$$

i. e., *the shearing force is equal to the derivative of the moment with respect to abscissa x.* This conclusion is also valid for a distributed load.

We shall derive a second important relation.

Let the beam be acted on by a continuously distributed uniform load of intensity q. A continuously distributed load is considered positive when it is directed upward. If the shearing force at any section of the beam is Q, the shearing force at a section a distance dx from this section is $Q+dQ$, where

$$dQ = q\,dx;$$

consequently,

$$q = \frac{dQ}{dx}. \tag{8.3}$$

Take the derivative of both sides of equality (8.2)

$$\frac{dQ}{dx} = \frac{d^2M}{dx^2}$$

or, taking into account relation (8.3), we obtain

$$q = \frac{d^2M}{dx^2}, \tag{8.4}$$

i. e., the second derivative of the bending moment with respect to abscissa is equal to the intensity of distributed load.

Relations (8.2) and (8.4) were obtained by the Russian scientist and engineer D. I. Jourawski.

61. Construction of Bending Moment and Shearing Force Diagrams

The normal and shearing stresses occurring on cross sections of a beam depend, respectively, on the magnitudes of bending moments M and shearing forces Q. Therefore, to determine the most dangerous sections, i. e., sections on which the maximum stresses occur, it is necessary to know the variation of moments and

shearing forces along the length of the beam. For greater clarity, these variations of M and Q along the length of the beam are usually represented graphically. These graphs showing the variation of M and Q are termed *bending moment and shearing force diagrams.* They are plotted in exactly the same manner as we plotted twisting moment diagrams for shafts. Laying off to a certain scale the magnitudes of bending moments acting at different sections from an axis parallel to the axis of the beam and joining the tops of the segments drawn, we obtain a bending moment diagram. To plot a shearing force diagram, we lay off segments which represent to a certain scale the magnitudes of shearing forces at different sections of the beam. In plotting bending moment and shearing force diagrams it is customary to lay off positive M and Q upward from the axis, and negative downward.

Relations (8.2) and (8.3) derived in the preceding section may be used in plotting M and Q diagrams. Indeed, the relation between bending moment and shearing force $Q=\frac{dM}{dx}$ may be given a geometrical interpretation. As is known, the derivative can be represented geometrically as the slope of the tangent to a curve at a given point, i. e.,

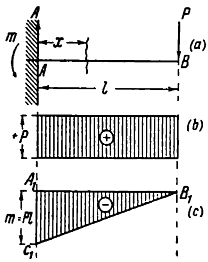


Fig. 115

$$Q=\frac{dM}{dx}=\tan\alpha.$$

Consequently, the shearing force at a given section may be regarded as the slope of the tangent to the moment diagram at the point corresponding to this section. At a section where $Q=\frac{dM}{dx}=0$, i. e., at a section where the shearing force passes through zero, the bending moment is maximum or minimum.

Further, at a section where the intensity of distributed load $q=\frac{dQ}{dx}=0$, the shearing force Q is maximum or minimum. This follows from the fact that when $q=0$ the tangent to the shearing force diagram is parallel to the axis of abscissas. On the basis of relation (8.2) it is possible to plot a moment diagram from the known shearing force diagram, and vice versa. However, the Q and M diagrams are plotted independently of each other, and relation (8.2) is used only for checking purposes.

We proceed to examples of plotting Q and M diagrams.

Let a beam fixed at one end be bent by a concentrated force

applied at its free end (Fig. 115 *a*). Construct a shearing force and a bending moment diagram.

(a) *Determination of Reactions.* The fixed support produces a vertical reaction and a reactive moment. We arbitrarily assume that the reaction A acts upward and the reactive moment counter-clockwise.

From the equilibrium conditions we have

$$\Sigma Y = 0, \qquad A - P = 0, \qquad A = P;$$
$$\Sigma M_A = 0, \qquad -m + Pl = 0, \qquad m = Pl.$$

We have obtained a positive sign for the reaction and the moment; this indicates that their senses were chosen correctly.

(b) *Construction of Shearing Force Diagram.* For any section of the beam the sum of the forces to the left of the section is expressed by a single force, namely the reaction at the fixed support, which is equal to P and directed upward. Therefore, the shearing force is constant over the whole length of the beam. To plot its diagram (Fig. 115*b*), we lay off upward a segment representing to scale the force P and draw a horizontal line.

(c) *Construction of Moment Diagram.* The bending moment at a section a distance x from the fixed end is found as the sum of the moments due to all loads to one side of the section. To the left of the section there act a reaction and a moment at the support; consequently, the moment at the section is

$$M = -m + Ax.$$

The reactive moment is taken with a minus sign because it rotates the left-hand part of the beam counter-clockwise, i. e., it bends the beam convex upward. The moment of the reaction A has the plus sign since it tends to rotate the left-hand part of the beam clockwise, i. e., it bends the beam convex downward. Substituting the values of m and A in the expression for the moment, we obtain

$$M = -Pl + Px = -P(l - x).$$

If we now write the moment for the same section due to the forces acting to the right of the section, which is simpler in the present case, we have

$$M = -P(l - x).$$

As might be expected, we have obtained the same value of the moment as earlier. The force P acting at the free end of the beam produces a moment tending to rotate the right-hand part of the beam clockwise, i. e., it bends the beam convex upward; therefore, the moment due to this force is negative. The expression for the bending moment obtained above should be regarded as an equation giving the law of variation of the magnitude of the moment with

the co-ordinate x of the section. The moment equation is a first-degree equation in this case; consequently, this is an equation of a straight line. Let us plot a bending moment diagram for the beam under consideration. Since the law of variation of moments represents a straight line in this case, it is necessary to know the moments for any two sections of the beam in order to plot a moment diagram.

We take a section coinciding with the plane of fixing, $x=0$. The moment for this section is numerically the largest. This is clear from the moment equation. This moment is given by

$$M_{\max} = -Pl.$$

For $x=l$, i. e., at the point of application of the bending force P

$$M = -P(l-x) = -P(l-l) = 0.$$

Having the moments for two sections, we plot a moment diagram, such as shown in Fig. 115*c*. The moments are laid off from the horizontal line A_1B_1 downward because the bending moments are negative. The moment diagram is thus represented by triangle $A_1B_1C_1$ in this case. The dangerous section of this beam is the section at the wall since the bending moment is maximum there.

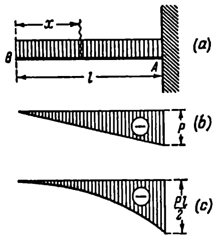


Fig. 116

Note that in this case the moment and shear diagrams could have been plotted in a simpler way without determining the reactions at the fixed support. To do this, it would be necessary to determine M and Q for a section a distance x from the fixed end, considering the forces lying to the right of the section.

It might be well to point out the following important rule: if a beam is subjected to concentrated forces and concentrated moments, the bending moment diagram is represented by straight line segments (parallel to the axis of the beam or sloping). To plot a moment diagram it is therefore sufficient in this case to calculate the moments only for certain sections of the beam which lie at the boundaries of separate portions of the diagram.

Consider another example. A beam of length l is fixed at one end and carries a load P uniformly distributed over its entire length, the magnitude of the load per unit length (the load intensity) being q (Fig. 116*a*). Construct a shearing force and a moment diagram.

(a) *Construction of Shearing Force Diagram.* We shall plot diagrams for this beam without determining reactions. The sum of all forces

lying to the left of section x is

$$Q = -qx.$$

From this equation it is seen that the shearing force varies according to a linear law.

At $x=0$,

$$Q=0;$$

at $x=l$,

$$Q = -ql = -P.$$

The maximum shearing force occurs at the fixed support

$$Q_{max} = -P;$$

the Q diagram is drawn using these two values (Fig. 116*b*).

(b) *Construction of Moment Diagram.* Considering the left-hand cut-off part, we obtain

$$M = -qx\frac{x}{2} = -\frac{qx^2}{2}.$$

The law of variation of the bending moment expressed by the last equation is the same over the whole length of the beam; x varies between the limits $x=0$ and $x=l$. The moment equation represents a parabola. To plot it we determine the moments for several sections:

at $x=0$,

$$M=0;$$

at $x=\frac{l}{2}$,

$$M = -\frac{ql^2}{8} = -P\frac{l}{8};$$

at $x=l$,

$$M = -\frac{ql^2}{2} = -P\frac{l}{2}.$$

The maximum moment occurs at the fixed support, i. e., at $x=l$

$$M_{max} = -\frac{ql^2}{2} = -\frac{Pl}{2}. \tag{8.5}$$

Comparing this moment with the maximum bending moment in the case of a concentrated force applied at the end of the beam, we see that the maximum bending moment for a uniformly distributed load is half its value for a concentrated force.

The bending moment diagram is drawn in Fig. 116*c* using the points of a parabola found above.

Example 53. Construct a shearing force and a bending moment diagram for the beam shown in Fig. 117*a*. The load intensity is

$q=2$ tons/m, the force $P=2$ tons, the concentrated moment at the end of the beam $m=3$ tons-m. The lengths of portions are $a=1$m, $b=3$m, $c=2$m.

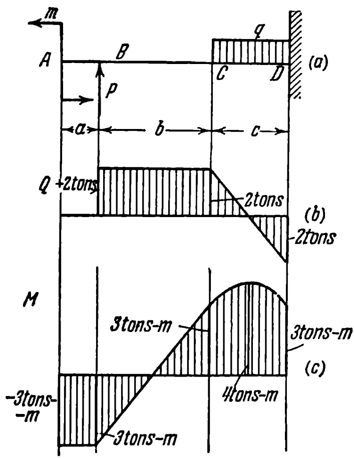


Fig. 117

Construction of Shearing Force Diagram. There is no shearing force in the first portion. In the second portion the shearing force is

$$Q_2 = P = 2 \text{ tons},$$

i. e., it is constant throughout the portion.

In the third portion the shearing force is

$$Q_3 = P - q[x-(a+b)] = 2-2(x-4)$$

or

$$Q_3 = 10-2x,$$

i. e., the shearing force varies linearly in the third portion.

Find two values of Q in this portion:

at $x=a+b=4$m,

$$Q = 10-2\times 4 = 2 \text{ tons};$$

at $x=a+b+c=6\text{m}$,

$$Q=10-2\times 6=-2 \text{ tons}.$$

The diagram is drawn using these values of Q (Fig. 117*b*).

Construction of Bending Moment Diagram. The moment in the first portion of the beam is constant and equal to

$$M_1=-m=-3 \text{ tons-m}.$$

In the second portion

$$M_2=-m+P(x-a)=-3+2(x-1)=-5+2x.$$

The bending moment in this portion varies linearly; to plot a diagram we find two values of the moment:

at $x=a=1$ m,

$$M_B=-5+2\times 1=-3 \text{ tons-m};$$

at $x=a+b=4$ m,

$$M_C=-5+2\times 4=3 \text{ tons-m}.$$

In the third portion

$$M_3=-m+P(x-a)-\frac{q[x-(a+b)]^2}{2}$$

or

$$M_3=-3+2(x-1)-\frac{2(x-4)^2}{2}=-x^2+10x-21.$$

The graph of the moment in this portion is a parabola; therefore, to plot a moment diagram we find three values of the moment:

at $x=a+b=4\text{m}$,

$$M_C=-4^2+10\times 4-21=3 \text{ tons-m};$$

at $x=a+b+c=6\text{m}$,

$$M_D=-6^2+10\times 6-21=3 \text{ tons-m};$$

at $x=a+b+\frac{c}{2}=5\text{m}$,

$$M=-5^2+10\times 5-21=4 \text{ tons-m}.$$

The diagram is drawn using the values of the moment found above (Fig. 117*c*).

Example 54. A beam of length l is simply supported at its ends and carries a concentrated force P (Fig. 118*a*).

Construct a shearing force and a bending moment diagram.

(a) *Determination of Reactions at Supports.* Writing moment equations with respect to points A and B, we obtain: the sum of mo-

ments of external forces about point A is

$$\Sigma M_A = -Bl + Pa = 0,$$

$$B = P\frac{a}{l};$$

the sum of moments of external forces about point B is

$$\Sigma M_B = Al - Pb = 0,$$

$$A = P\frac{b}{l}.$$

The advantage of applying moment equations in both cases resides in the fact that each equation involves only one unknown reaction.

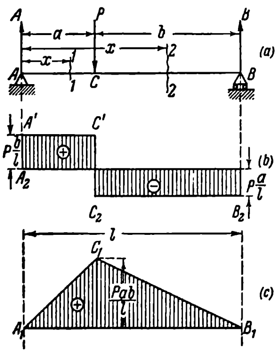


Fig. 118

To check the reactions found above we use another equilibrium condition, namely, $\Sigma Y = 0$

$$\Sigma Y = A + B - P = 0.$$

Substituting the values of A and B, we see that the reactions are correct.

(b) *Construction of Shearing Force Diagram.* The beam has two portions, the first portion extending from the left support A to the section where the force P is applied, i. e., to section C, and the second portion from section C to the right support B. For any section in the first portion, the sum of forces to the left of the section is expressed by a single force, viz. the reaction at the left support $A = P\frac{b}{l}$, which is directed upward. Therefore, the shearing force in the first portion is constant. To plot a diagram, we lay off from point A_2 (Fig. 118*b*) upward a segment A_2A' representing to scale the force $P\frac{b}{l}$ and draw a horizontal line $A'C'$ up to the end of the first portion. In passing through the boundary between the two portions, i. e., through the section where the force P is applied, the shearing force undergoes an abrupt change. The absolute value of this change is equal to the magnitude of the concentrated force applied at that section. Thus, the shearing force in the second portion is equal to

the sum of the forces

$$P\frac{b}{l}+(-P)=P\frac{b-l}{l}=-P\frac{a}{l}.$$

Consequently, the shearing force in the second portion is negative, remaining constant over the whole portion. To complete the construction of the shearing force diagram, we lay off from point C' downward a segment $C'C_2$ representing to scale the force P, and from point C_2 draw a horizontal line C_2B_2.

(c) *Construction of Moment Diagram.* The bending moment at any section in the first portion a distance x from the left support depends on one force A only and is expressed as

$$M_1=Ax=P\frac{b}{l}x. \qquad \text{(a)}$$

From this equation it is seen that the bending moment increases directly as the distance x. The variable x in this moment equation for the first portion can take values only between $x=0$ and $x=a$; larger values of x correspond to sections in the second portion of the beam, for which its own moment equation exists.

Since the equation (a) is of the first degree in x, the bending moment varies according to a linear law. Consequently, to plot a diagram in this portion it is sufficient to know the values of the moment for any two sections.

Over support A, i. e., at $x=0$, $M_A=0$. Under the force P, i. e., at $x=a$, $M_C=\frac{Pab}{l}$.

We now write the bending moment equation for the second portion; to do this we find, for a section a distance x from the left support, the bending moment due to all forces lying to the left of this section

$$M_2=Ax-P(x-a)=\frac{Pb}{l}x-P(x-a). \qquad \text{(b)}$$

This is an equation of the first degree; consequently, the bending moment in the second portion also varies according to a linear law. But, in distinction to the equation (a), the distance x in the equation (b) may vary within the limits of the second portion, i. e., from $x=a$ to $x=a+b=l$. We have

at $x=a$,

$$M_C=P\frac{b}{l}a-P(a-a)=\frac{Pab}{l};$$

at $x=l$, i. e., over the right support,

$$M_B=\frac{Pb}{l}l-P(l-a)=0,$$

Consequently, the dangerous or the design section of the beam corresponding to the maximum bending moment

$$M_{max} = \frac{Pab}{l} \tag{8.6}$$

is under the force P.

For the particular case when $a = b = \frac{l}{2}$, i.e., when the force P is applied at mid-length, we have

$$M_{max} = \frac{Pl}{4}. \tag{8.7}$$

The bending moments vary linearly in both portions of the beam. The moments over the supports are zero. Consequently, the moment diagram for the beam is represented by a broken line $A_1C_1B_1$ (Fig. 118c). The height of the shaded triangle $A_1C_1B_1$ represents to scale the moment $P\frac{ab}{l}$.

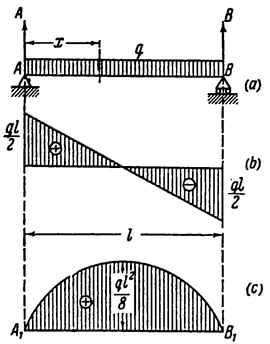


Fig. 119

At section C where the shearing force passes through zero, the bending moment has a maximum value.

Example 55. A beam of length l is simply supported at its ends and carries a uniformly distributed load of intensity q (Fig. 119a).

Construct a shearing force and a bending moment diagram.

(a) *Determination of Reactions.* The total load acting on the beam is ql. Since this load is uniformly distributed throughout the span of the beam, the reactions at the supports are

$$A = B = \frac{ql}{2}.$$

(b) *Construction of Shearing Force Diagram.* The beam has only one portion. The sum of forces lying to the left of any section is given by the equation

$$Q = \frac{ql}{2} - qx.$$

Since the shearing force equation is an equation of the first degree, the diagram is a straight line; therefore, it is sufficient to

know two values of the shearing force to draw this line:

at $x=0$,

$$Q_A=\frac{ql}{2}=A;$$

at $x=l$,

$$Q_B=-\frac{ql}{2}=-B.$$

The shearing force diagram is shown in Fig. 119*b*.

(c) *Construction of Moment Diagram.* Since the beam has only one portion, the law of variation of the bending moment is the same over the entire length of the beam. The bending moment at any section of the beam a distance x from the left support is equal to the sum of the moments due to the reaction $A=\frac{ql}{2}$ and the uniformly distributed load acting over the length x of the beam

$$M=\frac{ql}{2}x-qx\frac{x}{2}=\frac{qx(l-x)}{2}.$$

The above expression for moments is an equation of a parabola. To plot a diagram, we determine the moments for several sections

at $x=0$,

$$M_A=0;$$

at $x=\frac{1}{4}l$,

$$M_{l/4}=\frac{ql}{2}\frac{l}{4}-\frac{ql}{4}\frac{l}{2\times 4}=\frac{3}{32}ql^2;$$

at $x=\frac{1}{2}l$,

$$M_{l/2}=\frac{ql}{2}\frac{l}{2}-\frac{ql}{2}\frac{l}{2\times 2}=\frac{ql^2}{8}.$$

Since the beam is loaded symmetrically, there is no need to calculate the bending moments in the right-hand half of the beam; they can be immediately written as

at $x=\frac{3}{4}l$,

$$M_{3l/4}=\frac{3}{32}ql^2;$$

at $x=l$,

$$M_B=0.$$

The maximum bending moment occurs at the middle of the beam where the shearing force passes through zero; it is given by

$$M_{max}=\frac{ql^2}{8}. \tag{8.8}$$

Denoting the magnitude of the load acting on the beam by P, i.e., putting $ql = P$, we have

$$M_{\max} = \frac{Pl}{8}.$$

From comparison of this formula with formula (8.7) for the maximum bending moment in the case of a concentrated load applied at the centre of beam it is seen that *in the case of a uniformly distributed load the maximum bending moment is half its value for a concentrated force of the same magnitude.*

The bending moment diagram is drawn in Fig. 119*c* using the values of the moments found above.

Example 56. A beam CD of length l with overhanging ends (Fig. 120*a*) carries a uniformly distributed load of intensity q. Construct a shearing force and a bending moment diagram.

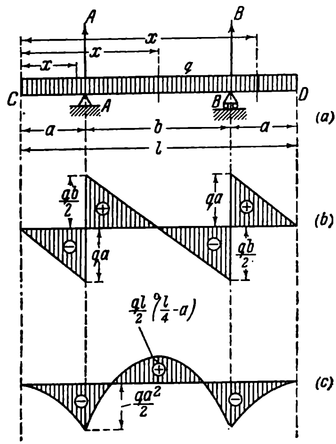


Fig. 120

(a) *Determination of Reactions.* Since the load is distributed symmetrically with respect to the supports of the beam, the reactions at the supports are

$$A = B = \frac{ql}{2}.$$

(b) *Construction of Shearing Force Diagram.* The shearing force at the end of the left overhang is zero; further it increases in absolute value up to support A, following the law

$$Q_1 = -qx.$$

The shearing force just to the left of support A attains a maximum negative value equal to $Q'_A = -qa$. Here the upward reaction $A = ql/2$ acts which is numerically larger than $Q'_A = -qa$. Therefore, the shearing force changes sign at the section over support A and becomes equal to

$$Q''_A = -qa + \frac{ql}{2} = -qa + \frac{q(2a+b)}{2} = \frac{qb}{2}.$$

Further the shearing force gradually decreases in the second portion due to the uniform load acting downward. For the section at the centre of the beam, the sum of forces acting to the left of this section is

$$-qa + \frac{ql}{2} - \frac{qb}{2} = -qa + \frac{q(2a+b)}{2} - \frac{qb}{2} = 0.$$

Consequently, the shearing force at the middle section of the beam is zero. For the section just to the left of support B, the sum of forces acting to the left of the support is

$$Q'_B = -q(a+b) + \frac{ql}{2} = -qa - qb + \frac{q(2a+b)}{2} = -\frac{qb}{2}.$$

Here the reaction $B = ql/2$ acts which is directed upward and is numerically larger than $Q'_B = -qb/2$. Hence, the shearing force changes sign over support B and becomes equal to

$$Q''_B = -\frac{qb}{2} + \frac{ql}{2} = qa.$$

Further the shearing force gradually decreases in the right overhang due to addition of the uniform load acting downward and becomes zero over the right end of the beam.

The shearing force diagram is shown in Fig. 120 *b*.

(c) *Construction of Moment Diagram.* The beam has three portions, CA, AB and BD.

The bending moment equation for the first portion is

$$M_1 = -qx\frac{x}{2} = -\frac{qx^2}{2}; \tag{a}$$

at $x = 0$,

$$M_C = 0;$$

at $x = a$,

$$M_A = -\frac{qa^2}{2}.$$

Since the equation (a) is an equation of a parabola, we have to take at least one more intermediate section to plot a moment diagram; for $x=\frac{a}{2}$, we obtain

$$M_{a/2}=-\frac{qa^2}{8}.$$

The bending moment equation for the second portion is

$$M_2=-\frac{qx^2}{2}+\frac{ql}{2}(x-a); \qquad \text{(b)}$$

at $x=a$,

$$M_A=-\frac{qa^2}{2};$$

at $x=a+b$,

$$M_B=-\frac{q(a+b)^2}{2}+\frac{ql(a+b-a)}{2};$$

but $l=2a+b$, therefore

$$M_B=-\frac{q(a^2+2ab+b^2)}{2}+\frac{q(2ab+b^2)}{2}=-\frac{qa^2}{2},$$

i. e., the moment over support B is equal to the moment over support A, a fact which might, of course, be predicted since, when a beam is subjected to symmetrical loading, the moment diagram must also be symmetrical.

The bending moment at the middle of the beam, i. e., at $x=l/2$ is, from the equation (b),

$$M_{l/2}=-\frac{ql^2}{8}+\frac{ql}{2}\left(\frac{l}{2}-a\right)=\frac{ql^2}{8}-\frac{qla}{2}=\frac{ql}{2}\left(\frac{l}{4}-a\right).$$

Example 57. A simply supported beam of length l is bent by a moment m (Fig. 121*a*). Construct a shearing force and a bending moment diagram.

(a) *Determination of Reactions at Supports.* We arbitrarily assume that the reaction A is directed upward and the reaction B downward; from equilibrium conditions we have

$$\Sigma M_B=0, \quad Al-m=0,$$

$$A=\frac{m}{l};$$

$$\Sigma Y=0, \quad A-B=0,$$

$$B=A=\frac{m}{l}.$$

Consequently, the reactions at the support caused by the applied moment are equal in magnitude but opposite in sense. The magni-

tude of these reactions is such that they produce a moment balancing the moment applied to the beam.

(b) *Construction of Shearing Force Diagram.* For any section in the first or second portion of the beam, the sum of forces lying to the left of the section is expressed by a single force, namely the reaction at support A, i. e., by the force m/l.

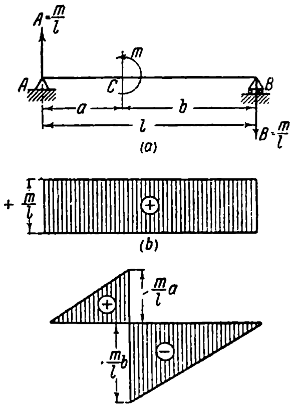
In the case considered, therefore, the shearing force diagram is a straight line parallel to the axis of the beam (Fig. 121b).

(c) *Construction of Moment Diagram.* The beam has two portions, the first portion extending from support A to the section where the external moment is applied, i. e., to point C, the second portion from point C to support B. The bending moment at any section in the first portion is expressed by

$$M_1 = Ax = \frac{m}{l}x.$$

Over support A, i. e., at $x=0$,

$$M_A = 0.$$



(c)

Fig. 121

At the section where the moment is applied, i. e., at $x=a$,

$$M_C = \frac{m}{l}a.$$

Write the moment equation for the second portion of the beam

$$M_2 = Ax - m = \frac{m}{l}x - m = \frac{m}{l}(x-l).$$

We have:

at $x=a$,

$$M_C = \frac{m}{l}(x-l) = m\frac{a-l}{l} = -\frac{m}{l}b;$$

at $x=a+b$,

$$M_B = \frac{m}{l}(a+b-l) = 0.$$

The moment diagram plotted from the values of the moments found above is shown in Fig. 121 c. The line segments of the mo-

ment diagram in the portions divided by the concentrated moment must be parallel by virtue of relation (8.2).

At the section where the external moment is applied, a discontinuity or "jump" in the moment diagram occurs and the diagram changes sign. Since $b > a$, the maximum moment is

$$M_{\max} = -\frac{m}{l} b. \tag{8.9}$$

In the particular case when the moment is applied at mid-span, i. e., when $a = b = l/2$

$$M_{\max} = \pm \frac{m}{2}.$$

Example 58. For the beam shown in Fig. 122*a* construct a shearing force and a moment diagram if $P = 2$ tons, $a = 0.5$ m.

(*a*) *Determination of Reactions.* The reactions A and B at the supports are found from the moment equations about points A and B

$$\sum M_A = -P\frac{a}{2} - Pa - B \times 2a + P \times 3a = 0,$$
$$B = 0.75\ P = 0.75 \times 2 = 1.5 \text{ tons};$$
$$\sum M_B = -P \times 2.5a + A \times 2a - Pa + Pa = 0,$$
$$A = 1.25P = 1.25 \times 2 = 2.5 \text{ tons}.$$

(*b*) *Construction of Shearing Force Diagram* (Fig. 122 *b*).
The beam has four portions, CA, AE, EB and BD. The shearing force at any section in the first portion a distance x from the left end C is

$$Q_1 = -\frac{P}{a}x = -\frac{2}{0.5}x = -4x.$$

The diagram in this portion is a straight line. We have:
at $x = 0$,

$$Q_C = 0;$$

at $x = a$,

$$Q_A = -4a = -4 \times 0.5 = -2 \text{ tons}.$$

For any section in the second portion, the sum of forces lying to the left of the section is

$$Q_2 = -P + A = -2 + 2.5 = 0.5 \text{ ton}.$$

Over support A, the shearing force changes sign from negative to positive. The shearing force in the second portion is constant over the entire portion, therefore the diagram is a straight line parallel to the axis.

The sum of forces lying to the left of any section in the third portion of the beam is the same as in the second portion, i. e.,

$$Q_3 = 0.5 \text{ ton}.$$

Therefore, the diagram in this portion is the same straight line parallel to the axis as in the second portion.

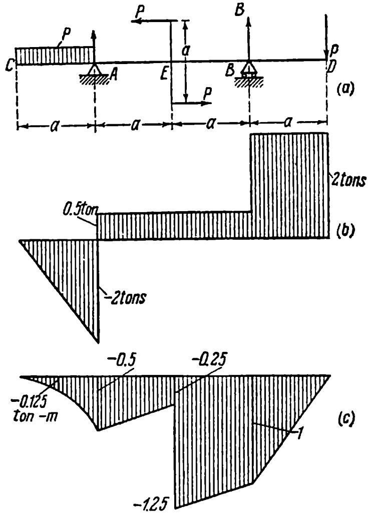


Fig. 122

The shearing force in the fourth portion is

$$Q_4 = -P + A + B = -2 + 2.5 + 1.5 = 2 \text{ tons}.$$

The diagram in this portion is also a straight line parallel to the axis. At the end of the fourth portion there is addition of a downward force of 2 tons, therefore the shearing force to the right of the end of the beam D is zero.

The maximum shearing force occurs over support A and throughout the fourth portion

$$Q_{max} = 2 \text{ tons}.$$

(c) *Construction of Moment Diagram* (Fig. 122*c*). The moment at any section in the first portion CA a distance x from the left end C of the beam is

$$M_1 = -\frac{P}{a} x \frac{x}{2} = -\frac{2}{0.5}\frac{x^2}{2} = -2x^2.$$

At $x=0$,

$$M_C = 0;$$

at $x=\frac{a}{2}$,

$$M = -2\left(\frac{a}{2}\right)^2 = -2\left(\frac{0.5}{2}\right)^2 = -0.125 \text{ ton-m};$$

at $x=a$,

$$M_A = -2a^2 = -2\times 0.5^2 = -0.5 \text{ ton-m}.$$

The moment in the second portion is

$$M_2 = -P\left(x-\frac{a}{2}\right) + A(x-a) =$$

$$= -2\left(x-\frac{0.5}{2}\right) + 2.5(x-0.5) = 0.5x - 0.75.$$

At $x=a$,

$$M_A = +0.5\times 0.5 - 0.75 = -0.5 \text{ ton-m};$$

at $x=2a$,

$$M_E = 0.5\times 2\times 0.5 - 0.75 = -0.25 \text{ ton-m}.$$

The moment in the third portion is

$$M_3 = -P\left(x-\frac{a}{2}\right) + A(x-a) - Pa =$$

$$= -2\left(x-\frac{0.5}{2}\right) + 2.5(x-0.5) - 2\times 0.5 = 0.5x - 1.75.$$

At $x=2a$,

$$M_E = 0.5\times 2\times 0.5 - 1.75 = -1.25 \text{ tons-m},$$

i. e., at the section where the external moment is applied the moment diagram has a discontinuity.

At $x=3a$,

$$M_B = 0.5\times 3\times 0.5 - 1.75 = -1 \text{ ton-m}.$$

The moment in the fourth portion due to forces lying to the right of the section is

$$M_4 = -P(4a-x) = -2(2-x).$$

At $x=3a$,

$$M_B = -2(2-1.5) = -1 \text{ ton-m};$$

at $x = 4a$,

$$M_D = 0.$$

The moment diagram is drawn using the values found above in Fig. 122c.

The maximum absolute value of the bending moment occurs at section E where the couple is applied

$$M_{max} = -1.25 \text{ tons-m}.$$

62. Check Questions

What is plane bending?

What is pure bending?

What happens to longitudinal fibres of material under bending?

What layer of fibres of a beam is called neutral?

What is the neutral axis?

What are the three types of support of beams?

What reactions arise in each of the three types of support of beams under the action of bending forces perpendicular to the axis of a beam?

Define the intensity of uniformly distributed load.

What is its dimension?

Define the bending moment and the shearing force at a given section.

What is the sign convention for the bending moment and the shearing force?

What is the relation between bending moment, shearing force and load intensity?

What is the purpose of plotting bending moment and shearing force diagrams?

Chapter IX

Stresses in bending and design of beams for strength

63. Determination of Normal Stresses in Bending

As in the case of any other type of deformation, the method of sections enables one to determine the bending moment and the shearing force at any section of a beam subjected to bending. The problem of distribution of elastic forces over the section is,

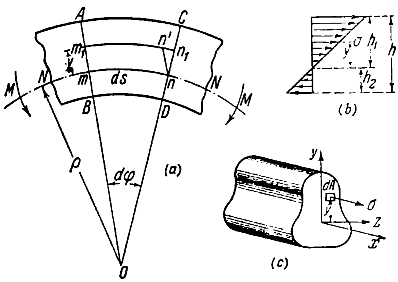


Fig. 123

in general, a statically indeterminate problem. As we saw above, such problems are solved by consideration of deformations. In the case of tension and compression, it was assumed that all fibres of material developed the same strains in the direction of the applied forces; from this it was concluded that the stresses were uniformly distributed over the section. The problem of distribution of stresses induced by torsion was solved on the assumption that the shearing strains of individual elements of a cross section were directly proportional to their distance from the axis of a bar. The law of distribution of stresses over a section in the case of bending can also be determined only by consideration of deformations.

Take part of a beam bent by two equal and opposite moments acting in the longitudinal plane of symmetry of the beam (Fig. 123*a*).

For clarity, the bending of the beam is greatly exaggerated in the figure. Actually, as in the case of any other type of deformation, we assume that the magnitude of bending deformation is very small and the deflected axis of the beam differs only slightly from the original straight axis.

Assume that line N-N is the neutral layer above which the fibres of the rod are stretched, and below compressed. Let the common centre of curvature of bent fibres be at point O, the radius of curvature of the neutral layer being ρ. Isolate an element $ABDC$ from the part of the rod under consideration by means of two very close sections passing through the centre of curvature of the rod and inclined to each other at an infinitesimal angle $d\varphi$. The infinitesimal length ds of a fibre mn of the neutral layer remains the same after bending. A fibre m_1n_1 a distance y from the neutral layer undergoes extension. In order to find the elongation, we draw from point n a line parallel to AB. The arc $n'n_1$ gives this elongation. From the similarity of the large triangle Omn and the small triangle $nn'n_1$, we have

$$\frac{n'n_1}{ds}=\frac{y}{\rho}. \tag{a}$$

The left-hand side of this equality represents the unit elongation of the fibre m_1n_1 which had the length ds before deformation. Denoting the unit elongation of the fibre under consideration by ε, we rewrite the equality (a) as

$$\varepsilon=\frac{y}{\rho}. \tag{b}$$

For any given section, the radius of curvature ρ is a constant. Therefore, from the equation (b) it may be concluded that the strains in the fibres of a bent rod are directly proportional to their distance from the neutral layer. Since the fibres of a rod undergo only simple tension or compression when subjected to bending, the distribution of elastic forces over the section can be determined using Hooke's law for tension and compression

$$\sigma=E\varepsilon. \tag{c}$$

Substituting in (c) the value of ε from the expression (b), we obtain

$$\sigma=E\frac{y}{\rho}. \tag{9.1}$$

Formula (9.1) obtained by consideration of deformations gives a law of distribution of elastic forces over the cross section. From this formula it follows that *the stresses on any cross section of a bent beam are directly proportional to the distance of the point under consideration from the neutral layer*. All the fibres equally distant from

the neutral layer have the same stresses, i. e., the stresses do not vary across the width of the beam.

For the neutral layer, $y=0$. Consequently, for this layer $\sigma=0$. In passing the neutral layer the sign of y changes and so does the sign of the stress σ. The maximum stresses on the section occur at points for which the distance y is maximum, i. e., at the top and bottom layers of the section.

The stress diagram for a cross section is shown in Fig. 123*b* where the tensile stresses are directed one way and the compressive stresses in the opposite way.

Formula (9.1) involves the radius of curvature of the neutral layer; to determine it, we isolate from the cross-sectional area (Fig. 123*c*) an elementary area dA a distance y from the neutral line. The elementary normal force acting on this area is, from (9.1),

$$dN=\sigma dA=\frac{Ey}{\rho}dA. \tag{d}$$

Since, on the basis of the equilibrium condition, all elastic forces acting at the section must give only a moment equal to the external moment, the sum of their projections on the beam axis x must be zero, i. e.,

$$\int_A \frac{Ey}{\rho}dA=0 \quad \text{or} \quad \frac{E}{\rho}\int_A y\,dA=0.$$

The ratio $E/\rho\neq 0$, consequently,

$$\int_A y\,dA=0.$$

This integral represents the static moment of the cross-sectional area with respect to the neutral line. If the static moment is zero, the axis with respect to which it is taken passes through the centroid of the section. A very important conclusion may be drawn from this, namely that *the neutral axis passes through the centroid of the cross section.*

The elementary moment of the internal force acting on the area dA about the neutral axis z is, from the equation (d),

$$dN\,y=\frac{E}{\rho}y\,dA\,y=\frac{E}{\rho}y^2\,dA.$$

The sum of all elementary moments of internal elastic forces must be equal to the external moment, from the equilibrium condition, i. e.,

$$\int_A \frac{E}{\rho}y^2\,dA=\frac{E}{\rho}\int_A y^2\,dA=M. \tag{e}$$

The integral $\int_A y^2\,dA$ represents the moment of inertia of the cross section with respect to the neutral axis. Denoting it by I, we obtain

$$\frac{E}{\rho} I = M$$

or, rewriting this expression in an alternate form, we have

$$\frac{1}{\rho} = \frac{M}{EI}. \tag{9.2}$$

Formula (9.2) is the fundamental formula in the flexure theory. The quantity $1/\rho$ *(the curvature of the deflected axis of the beam)* characterizes the magnitude of bending deformation; from formula (9.2) it follows that the bending deformation is directly proportional to the bending moment and inversely proportional to the *product EI* called the *flexural rigidity of the beam*. Moreover, from the same formula it may also be concluded that portions of a beam subjected to a constant moment, i. e., portions which are in a state of pure bending, are bent to a circular arc of radius EI/M.

After determining ρ from formula (9.2) and substituting its value in Eq. (9.1), we obtain

$$\sigma = \frac{My}{I}. \tag{9.3}$$

From this equation, as from Eq. (9.1), it is seen that the maximum stresses occur in the fibres most remote from the neutral axis. In strength design problems we are generally interested in the maximum stresses. Consequently, y should be replaced in Eq. (9.3) by the distance of the outermost fibres from the neutral axis. If, as in our case, the centroid of the cross section is not at mid-depth, the maximum stresses for the outermost fibres of the section are

$$\sigma_{\max_1} = \frac{Mh_1}{I}, \quad \sigma_{\max_2} = -\frac{Mh_2}{I}. \tag{9.4}$$

One of these stresses is the maximum tensile stress in the stretched fibres, and the other the maximum compressive stress in the compressed fibres. If the beam material is of equal strength in tension and compression, it is sufficient to determine only one maximum stress for the fibres most remote from the neutral axis regardless of whether they are stretched or compressed.

In our case the maximum stress is

$$\sigma_{\max} = \frac{Mh_1}{I},$$

since $h_1 > h_2$.

If a material, such as cast iron, has different strength in tension and compression, it is necessary to determine both maximum stres-

ses. If the centroid of the section of a rod made of material of equal strength in tension and compression is at mid-depth, i. e., $h_1 = h_2 = h/2$, the tensile and compressive stresses are numerically equal in this case and are given by

$$\sigma_{max} = \frac{M\frac{h}{2}}{I}, \tag{9.5}$$

where $h/2$ is the distance of the outermost stretched or compressed fibre from the neutral axis.

If the magnitude of the moment varies along the length of the rod, the maximum stresses are determined at a section where the bending moment is maximum. This section of the rod has been previously referred to as the *dangerous section*.

The ratio of the moment of inertia I to the distance y_{max} of the outermost fibre from the neutral line is called the *section modulus* and denoted by Z

$$\frac{I}{y_{max}} = Z. \tag{9.6}$$

Since the dimension of I is cm^4 and that of y_{max} is cm, the dimension of Z is cm^3.

Taking into account expression (9.6), formula (9.5) for determining the maximum stress in a beam for which $y_{max} = h_1 = h_2 = h/2$ may be rewritten as

$$\sigma_{max} = \frac{M_{max}}{Z}. \tag{9.7}$$

Denote the section moduli of a rod for which $h_1 \neq h_2$ by Z_1 and Z_2, respectively,

$$\frac{I}{h_1} = Z_1, \qquad \frac{I}{h_2} = Z_2.$$

We have then

$$\sigma_{max_1} = \frac{M_{max}}{Z_1}, \qquad \sigma_{max_2} = -\frac{M_{max}}{Z_2}. \tag{9.8}$$

In deriving the formulas of this section we assumed that the beam had a longitudinal plane of symmetry and that the bending deformation took place in that plane. Consider now the bending of a beam with an unsymmetrical cross section, such as an unequal angle (Fig. 124).

Suppose that the external loads act in a principal plane of the beam, i. e., in a longitudinal plane passing through one of the principal axes of the cross section. Let the y and z axes be principal centroidal axes of the section, and the bending moments act in a principal plane passing through the y axis. Let us see whether the equilibrium conditions are satisfied if we assume that the stress

distribution over the section of the beam in this case is the same as in the case of the bending moments acting in a plane of symmetry of the rod, i. e., as given by formula (9.3), and whether the neutral line coincides with the principal centroidal z axis of the cross section of the rod.

If the stress distribution over the cross-section follows formula (9.3), the moment of the internal forces about the neutral axis

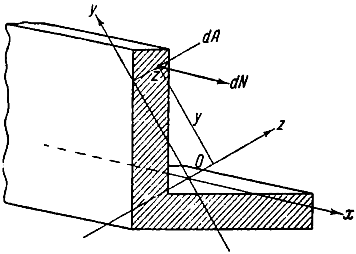


Fig. 124

balances the external bending moment. The elementary force acting on any area dA with coordinates z, y is, from formula (9.3),

$$dN = \sigma\, dA = \frac{My}{I}\, dA.$$

The elementary moment due to this force about the principal axis Oy is

$$dM_y = \frac{My}{I}\, dA\, z.$$

The sum of the elementary moments of all internal forces acting on the cross section about the principal axis Oy of the rod is

$$M_y = \int\limits_A \frac{My}{I} z\, dA = \frac{M}{I} \int\limits_A yz\, dA.$$

The external bending moment about the same axis of the rod is zero, consequently M_y must also be zero, and this is possible only if $\int\limits_A yz\, dA = 0$, i. e., if the product of inertia of the section is zero. Since the y and z axes are principal axes, the product of inertia with respect to these axes is truly zero (see Sec. 53). Thus, if the bending moment acts in one of the principal planes of the

beam, the other principal plane coincides with the plane of the neutral layer, or, in other words, the neutral line of the section coincides with the principal centroidal axis of the section, and *the stress distribution over the section is the same as when bending moments act in a plane of symmetry of the rod.*

64. Section Moduli for Common Sections

Knowing the axial moment of inertia of the cross-sectional area with respect to the neutral line and the distance of the outermost fibres from the neutral line, we determine the section modulus by the general formula (9.6)

$$Z=\frac{I}{y_{\max}}.$$

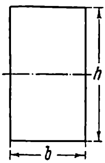


Fig. 125

The section modulus for a rectangular section of base b and height h (Fig. 125) is

$$Z=\frac{I}{\frac{h}{2}}=\frac{bh^3}{12\frac{h}{2}}=\frac{bh^2}{6}. \qquad (9.9)$$

In formula (9.7), the section modulus appears in the denominator, consequently, the strength of a beam increases with increase in the section modulus. Therefore, from the point of view of saving of material the most rational sections are those which have large section moduli and small areas. Thus, a rectangular section of a beam in which $h>b$ is more economical than a square one. Indeed, the section modulus for a square section is

$$Z_s=\frac{I}{\frac{a}{2}}=\frac{a^4}{12\frac{a}{2}}=\frac{a^3}{6}. \qquad (9.10)$$

If the cross-sectional areas of the rectangular and square sections are equal, i.e.,

$$bh=a^2,$$

the ratio of the section modulus for the rectangular section to that for the square section is

$$\frac{Z_r}{Z_s}=\frac{bh^2}{6}:\frac{a^3}{6}=\frac{bh}{a^2}\frac{h}{a}=\frac{h}{a}.$$

Since $bh=a^2$ and $h>b$, $h>a$; consequently, Z_r for the rectangular section is larger than Z_s for the square section of equal area, and they bear the same ratio as the height of the rectangle and the side of the square.

A beam of rectangular section placed flat has the section modulus

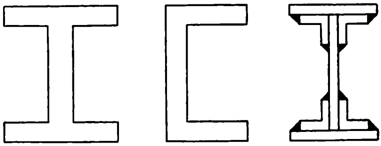
$$Z=\frac{hb^2}{6}. \qquad (9.11)$$

From comparison of formulas (9.9) and (9.11) it is seen that in the latter case (beam placed flat) the section modulus is smaller; consequently, it is not advantageous to place a beam flat. This can easily be verified by bending a usual ruler.

The section modulus for a circular section is

$$Z = \frac{I}{\frac{d}{2}} = \frac{\pi d^4}{64\frac{d}{2}} = \frac{\pi d^3}{32} \cong 0.1d^3. \tag{9.12}$$

Rectangular and circular sections are most commonly used in wooden beams. For metal beams, other sections are chosen which are more economical than rectangular and circular sections.



(a) (b) (c)

Fig. 126

Since the material is only slightly stressed near the neutral axis, it is advantageous to put more material farther away from the neutral axis, i. e., to transfer it from places where it is only slightly stressed to places where it will be more stressed. Therefore, for beams made of metal having the same strength in tension and compression, sections are often chosen in the form of an I-section (Fig. 126*a*), channel (Fig. 126*b*); welded beams are in common use (Fig. 126*c*). Such beams provide considerably larger section moduli than beams of rectangular and circular section having the same cross-sectional area. The neutral line passes through the middle of the height in these sections, therefore the maximum tensile and compressive stresses for such sections are the same. For beams whose material has different strength in tension and compression, such as cast iron, sections are made unsymmetrical with respect to the neutral line, as in a T-section (Fig. 127). A T-beam is then placed so that there will be tensile stresses in the horizontal flange; since the neutral axis is shifted to the horizontal flange, tensile stresses are smaller than compressive stresses (see Sec. 65).

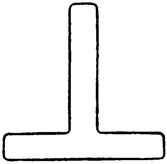


Fig. 127

In practice an annular (tubular) section is not uncommon. The section modulus for an annular section is larger than that for a circular section of the same area since the material is more rationally utilized in the annular section: it is removed farther away from the neutral line.

The section modulus for an annular section of outer diameter D and inner diameter d is

$$Z=\frac{I}{\frac{D}{2}}=\frac{\pi(D^4-d^4)}{64\frac{D}{2}}=\frac{\pi(D^4-d^4)}{32D}\cong 0.1\,\frac{D^4-d^4}{D}, \qquad (9.13)$$

Putting $d/D=\alpha$, we obtain an alternate expression for Z for an annular section

$$Z\cong 0.1D^3(1-\alpha^4). \qquad (9.14)$$

Expression (9.14) is usually used to determine the diameters D and d of a section when the ratio $d/D=\alpha$ is given.

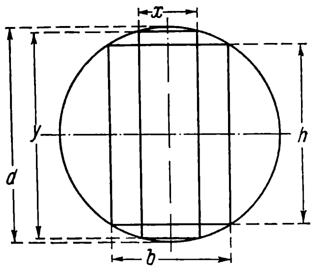


Fig. 128

Moments of inertia and section moduli for rolled sections of standardized sizes are given in tables of GOST (State Standard of the USSR).

Example 59. Determine the ratio between the sides h and b of a rectangular beam cut out from a round log of diameter d (Fig. 128) so that the section modulus Z will be maximum.

Solution. The section modulus for a rectangle of base x and height y is

$$Z=\frac{xy^2}{6};$$

since

$$y=\sqrt{d^2-x^2},$$

we have

$$Z=\frac{x(d^2-x^2)}{6}=\frac{xd^2-x^3}{6}.$$

To determine the maximum value of Z we set up the expression $dZ/dx=0$

$$\frac{dZ}{dx}=\frac{d^2}{6}-\frac{3x^2}{6}=0,$$

whence

$$3x^2=d^2$$

and consequently

$$x=\frac{d}{\sqrt{3}}.$$

The height of a rectangle of base $d/\sqrt{3}$ is

$$y=\sqrt{d^2-x^2}=\sqrt{d^2-\frac{d^2}{3}}=d\sqrt{\frac{2}{3}}.$$

Thus, the ratio of the sides of a rectangle cut out from a circle and having the maximum section modulus is

$$y:x=d\sqrt{\frac{2}{3}}:\frac{d}{\sqrt{3}}=\sqrt{2}$$

or

$$h:b=\sqrt{2}\cong 7:5.$$

Example 60. Determine how the section modulus with respect to a diagonal AB of a square section of side a (Fig. 129) placed on its rib is changed if angles with lateral sides equal to 1/9 of the side of the square are cut off at the top and bottom. The cut-off angles are shown by hatching in the drawing.

Solution. The section modulus of the complete square with respect to diagonal AB is

$$Z=\frac{I}{h}=\frac{a^4}{12\frac{a\sqrt{2}}{2}}=\frac{a^3\sqrt{2}}{12}.$$

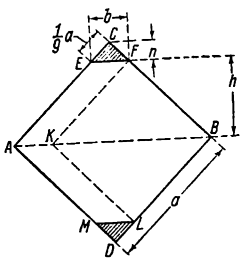


Fig. 129

Determine now the section modulus of the part of the square remaining after the triangles are removed. This part may be regarded as consisting of square $FKLB$ and two parallelograms, $AEFK$ and $AKLM$.

The base of the triangle is

$$b=\frac{1}{9}a\sqrt{2}.$$

The height of the triangle is

$$h=\frac{1}{18}a\sqrt{2}.$$

The moment of inertia of square $FKLB$ with respect to the diagonal is

$$I'=\frac{\left(a-\frac{1}{9}a\right)^4}{12}.$$

The sum of the moments of inertia of parallelograms $AEFK$ and $AKLM$ put base to base is

$$I''=\frac{2\frac{1}{9}a\sqrt{2}h^3}{3}=\frac{2a\sqrt{2}\left(\frac{\sqrt{2}}{2}a-\frac{\sqrt{2}}{18}a\right)^3}{27}=\frac{8a^4\left(\frac{4}{9}\right)^3}{27}.$$

The section modulus of the remaining part of the square is

$$Z'=\frac{I'+I''}{h}=\left[\frac{\left(\frac{8}{9}a\right)^4}{12}+\frac{8a^4}{27}\left(\frac{4}{9}\right)^3\right]:\frac{4\sqrt{2}}{9}a=\frac{64\sqrt{2}}{729}a^3.$$

The ratio of the section modulus Z' to the section modulus of the complete square is

$$\frac{Z'}{Z}=\frac{64\sqrt{2}a^3}{729}:\frac{a^3\sqrt{2}}{12}\cong 1.054.$$

Thus, the removal of the triangles reduces the cross-sectional area but increases the section modulus by about 5 per cent. This is due to the fact that, though the moment of inertia of the square is reduced by removing the triangles, the depth of the beam is reduced to a still greater degree.

65. Design Flexure Formulas. Examples of Designing Beams

It has been shown above that, when a beam is bent by transverse forces, shearing forces act at cross sections of the beam in addition to bending moments which produce normal stresses. Shearing stresses induced by transverse forces are of considerable magnitude only in very short beams. Therefore, beams are usually designed only on the basis of normal stresses.

Knowing the allowable tensile $[\sigma_t]$ and compressive $[\sigma_c]$ stresses for a given material, we can write strength-design equations for

bending, using formulas (9.8)

$$\left.\begin{aligned} \sigma_1 &= \frac{M_{\max}}{Z_1} \leqslant [\sigma_t], \\ \sigma_2 &= \frac{M_{\max}}{Z_2} \leqslant [\sigma_c] \end{aligned}\right\} \qquad (9.15)$$

and requiring that the stresses in the extreme fibres not exceed the allowable tensile and compressive stresses.

If $[\sigma_t] = [\sigma_c] = [\sigma]$ for a given material, the design flexure formula becomes

$$\sigma_{\max} = \frac{M_{\max}}{Z} \leqslant [\sigma]. \qquad (9.16)$$

The strength-design equations for bending are similar to the design equations considered above. Again, they make it possible to solve three problems: (1) to determine stresses if the bending moment acting on a beam and the section modulus are known, (2) to determine the bending moment acting on a beam if the allowable stress and the section modulus are known, (3) to determine the section modulus and hence the dimensions of the section if the bending moment and the allowable stress are known.

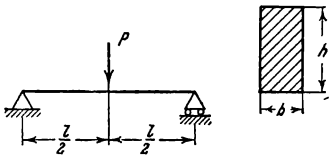


Fig. 130

If the beam material has the same strength in tension and compression, it is advisable to choose a cross section of the beam with two axes of symmetry. If, however, the allowable tensile and compressive stresses are different for a given material, it is more expeditious to choose a section that is not symmetrical about the neutral axis and to place the beam so that the fibres which undergo more dangerous stresses are nearer the neutral axis.

Example 61. Determine the permissible magnitude of a force P which bends a simply supported steel beam 1 m long (Fig. 130). The section of the beam is a rectangle, $b = 4$ cm, $h = 6$ cm, the yield strength of the material is $\sigma_y = 3{,}000$ kgf/cm^2, the factor of safety $k = 1.5$.

Solution. The maximum bending moment at mid-length is, from formula (8.8),

$$M_{max}=\frac{Pl}{4}.$$

Write the strength condition

$$\frac{M_{max}}{Z}\leqslant[\sigma]=\frac{\sigma_y}{k} \quad\text{or}\quad \frac{Pl}{4\frac{bh^2}{6}}\leqslant\frac{\sigma_y}{k},$$

whence

$$P\leqslant\frac{4\sigma_y bh^2}{6\times1.5l}=\frac{4\times3,000\times4\times6^2}{6\times1.5\times100}=1,920\text{ kgf}.$$

Example 62. Determine the stress on the design section AB of a gear tooth (Fig. 131) if the pressure $P=4,500$ kgf on the tooth is applied to its top. The height of the tooth is $h=30$ mm, its length $b=90$ mm, its thickness $S=22$ mm.

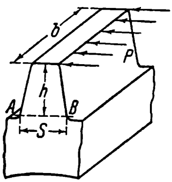


Fig. 131

Solution. The bending moment at the design section is

$$M=Ph=4,500\times3=13,500\text{ kgf-cm}.$$

The section modulus for the design section is

$$Z=\frac{bS^2}{6}=\frac{9\times2.2^2}{6}=7.26\text{ cm}^3.$$

The stress on the design section is

$$\sigma=\frac{M}{Z}=\frac{13,500}{7.26}=1,860\text{ kgf/cm}^2.$$

Example 63. Determine the diameter of a car axle if the forces acting at the ends of the axle (Fig. 132 *a*) are $P=2,000$ kgf, the distance of the points of application of the forces from the middle plane of the wheels is $a=15$ cm, the allowable stress $[\sigma]=1,000$ kgf/cm².

Solution. The reactions at supports A and B are

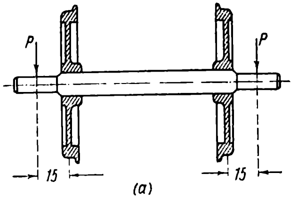
$$A=B=2,000\text{ kgf}.$$

The maximum bending moment (Fig. 132 *b*) is

$$M_{max}=Pa=2,000\times15=30,000\text{ kgf-cm}.$$

The section modulus is

$$Z=\frac{M_{max}}{[\sigma]}=\frac{30,000}{1,000}=30\text{ cm}^3$$



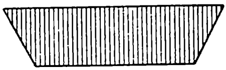


Fig. 132

and, since $Z=0.1\,d^3$,

$$d=\sqrt[3]{\frac{30}{0.1}}=6.69\cong 7\ \text{cm}.$$

66. Shearing Stresses in a Beam of Rectangular Section. Jourawski's Formula

In Sec. 56 we saw that, in the general case of bending, both bending moments (producing normal stresses) and shearing forces occur on cross sections of a beam. The shearing force tends to move one part of a beam with respect to another in a direction perpendicular to the axis of the beam. Therefore, the shearing force produces shearing stresses in the plane of the cross section. By virtue of the law of equal shearing stresses on orthogonal planes, shearing stresses are induced in a beam which act parallel to the neutral plane and tend to move horizontal layers of the beam with respect to one another. The existence of the latter stresses can easily be verified by the following simple experiment.

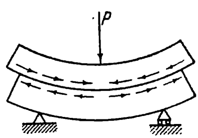


Fig. 133

Take a beam which is made up of two loose rods and apply a bending load to it, as shown in Fig. 133. Each rod will behave as a separate beam, the top fibres of the rods will be compressed, and the bottom fibres stretched. The experiment shows that the ends of this composite beam assume a stepped position during bending, i. e., the separate rods slip with respect to each other in a

longitudinal direction. In an integral beam the ends do not exhibit a stepped shape. Obviously, in this case the elastic forces arising in longitudinal layers of the beam prevent this longitudinal slip. In Fig. 133 these tangential forces are indicated by arrows. The existence of longitudinal slip explains, in particular, the occurrence of longitudinal cracks in beams whose material, such as wood, is weak in shear parallel to the grain. Now that the existence of shearing stresses under bending is established, we proceed to the

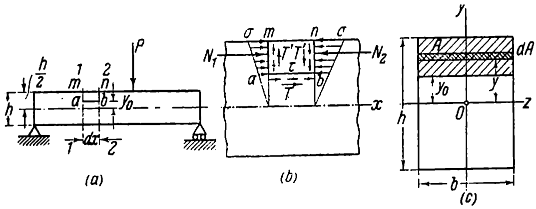


Fig. 134

determination of their magnitude and their law of distribution across the depth of a beam. We shall consider the simplest case when the beam has a rectangular section. In the case of a rectangular section it may be assumed that the shearing stresses on the cross section are parallel to the shearing force Q and that their magnitude is constant across the width of the beam, i. e., throughout the neutral axis z-z. As exact investigations show, the error introduced by this assumption is very small.

Take a beam of rectangular section of sides $h \times b$ subjected to a force P (Fig. 134*a*). In the left-hand portion of the beam, pass two cross sections, *1-1* and *2-2*, a distance dx apart and a longitudinal section *ab* parallel to the neutral layer at a distance y_0 from the latter. These three sections through the beam cut out an infinitely narrow parallelepiped *mabn* of dimensions dx, $\frac{h}{2}-y_0$ and b. Denote the bending moment at section *1-1* by M and at section *2-2* by $M' = M + dM$. The cross sections *1-1* and *2-2* are passed in the left-hand portion of the beam where the bending moments and the shearing forces are positive. Therefore, $M' > M$ and $dM > 0$. Imagine the parallelepiped *mabn* to be isolated from the beam and consider the equilibrium conditions for it (Fig. 134*b*). The action of the removed parts of the beam on it is replaced by internal forces. The lateral faces of the parallelepiped formed by the planes of the sections are acted on by compressive normal forces N_1 and

N_2 produced by the bending moments, the force N_1 on the face *ma* being smaller than the force N_2 on the face *nb* since $M' > M$. Besides, the lateral faces are acted on by tangential forces produced by the shearing forces. The resultant of these forces is denoted by T'. Since $N_2 > N_1$, the parallelepiped must move to the left. This movement, however, is prevented by the tangential forces occurring on face *ab*. The resultant of these forces is denoted by T.

The elementary normal force acting on an infinitesimal area dA of the left-hand face of the parallelepiped (Fig. 134*c*), which is at a distance y from the neutral axis, is

$$dN_1 = \sigma\, dA = \frac{My}{I}\, dA,$$

where I is the moment of inertia of the entire section with respect to the neutral axis.

The normal force acting on the whole of the left-hand face of the parallelepiped is

$$N_1 = \int\limits_{A_y} \sigma\, dA = \int\limits_{A_y} \frac{My}{I}\, dA,$$

where A_y is the area of a part of the cross section from y_0 to $h/2$.

In the integrand the quantity M/I is constant as M and I are constant at a given cross section; therefore

$$N_1 = \frac{M}{I} \int\limits_{A_y} y\, dA. \tag{a}$$

Similarly we find the magnitude of the force N_2 acting on the right-hand lateral face of the parallelepiped

$$N_2 = \frac{M_1 + dM}{I} \int\limits_{A_y} y\, dA. \tag{b}$$

The magnitude of the resultant T of the tangential forces acting on the lower face of the parallelepiped, if the forces are assumed to be uniformly distributed along the infinitesimal length dx of the face, is

$$T = \tau b\, dx. \tag{c}$$

Projecting all forces acting on the parallelepiped on the x axis, we obtain

$$\sum X = 0, \qquad N_2 - N_1 = T. \tag{d}$$

Substituting in the expression (d) the values of N_1, N_2 and T from the expressions (a), (b) and (c), respectively, we obtain

$$\frac{M + dM}{I} \int\limits_{A_y} y\, dA - \frac{M}{I} \int\limits_{A_y} y\, dA = \tau b\, dx$$

or

$$\frac{dM}{I}\int_{A_y} y\,dA = \tau b\,dx,$$

whence

$$\tau = \frac{dM}{dx}\frac{1}{bI}\int_{A_y} y\,dA. \tag{e}$$

The integral $\int_{A_y} y\,dA$ represents the static moment of the shaded area with respect to the neutral axis, i. e., of the area of the lateral face of the parallelepiped. Designate it as

$$\int_{A_y} y\,dA = S$$

The quantity dM/dx is equal to the shearing force Q. Therefore, the expression (e) may be finally written as

$$\tau = \frac{QS}{Ib}. \tag{9.17}$$

Thus, the shearing stress in a longitudinal layer of a beam is equal to the product of the shearing force (Q) at the section in question and the static moment (S) with respect to the centroidal axis of a part of the cross section located above the level y_0 being considered, divided by the moment of inertia (I) of the whole section with respect to the neutral axis and by the width (b) of the cross section of the beam.

For any given section, the quantities Q and I are constant. Therefore, the shearing stresses vary directly as the ratio S/b. In the extreme top and bottom longitudinal layers of the beam, i. e., where the normal stresses due to the bending moment have maximum values, *the shearing stresses are zero* since $S = 0$ for them. For sections in which the width b remains constant throughout the section, the maximum shearing stresses occur in the neutral layer since the static moment is maximum for the neutral layer. In the general case the quantities S and b are variable. It is impossible to predict where the maximum shearing stresses will occur. It can only be said that they will be maximum for layers for which the ratio S/b has a maximum value.

By virtue of the law of equal shearing stresses, formula (9.17) determines also the magnitude of shearing stresses on transverse sections of a beam. Consequently, the shearing stresses are not uniformly distributed at a cross section of a beam.

The shearing stresses produce shearing deformation of a beam which, however, does not affect the distribution of normal stresses defined by formula (9.7). As a result of the shearing deformation

plane cross sections do not remain plane after bending, as in the case of pure bending, but warp. Figure 135 shows the warping of cross sections. Where the shearing stresses reach their maximum values, the shear is maximum; the fibres most remote from the neutral layer have no shearing stresses; therefore no shear occurs there and the curves mn remain perpendicular to the surfaces of the beam.

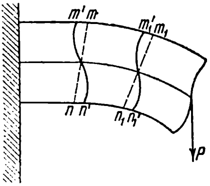


Fig. 135

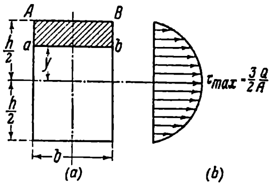


Fig. 136

Consider the distribution of shearing stresses in a beam of rectangular section. The shearing stresses along the line ab a distance y from the neutral axis (Fig. 136a) are, from formula (9.17),

$$\tau=\frac{QS}{Ib}.$$

For any given section of the beam, Q, I and b are constants. Therefore, τ varies directly as S. The static moment of the shaded rectangular area $AabB$ above line ab with respect to the neutral axis is

$$S=b\left(\frac{h}{2}-y\right)\left(y+\frac{\frac{h}{2}-y}{2}\right).$$

Here the product of the first two multipliers represents the area of rectangle $AabB$, and the third multiplier represents the distance of the centroid of this rectangle from the neutral axis. Rearranging terms in this expression, we find

$$S=\frac{b}{2}\left(\frac{h}{2}-y\right)\left(\frac{h}{2}+y\right)=\frac{b}{2}\left(\frac{h^2}{4}-y^2\right).$$

The moment of inertia of the whole section with respect to the neutral axis is

$$I=\frac{bh^3}{12}.$$

Substituting the values of S and I in formula (9.17), we obtain

$$\tau = \frac{Q\frac{b}{2}\left(\frac{h^2}{4}-y^2\right)}{\frac{bh^3}{12}b} = \frac{6Q}{bh^3}\left(\frac{h^2}{4}-y^2\right). \qquad (9.18)$$

The maximum value of the shearing stress occurs at the neutral axis where $y=0$

$$\tau_{\max} = \frac{6Q}{bh^3}\frac{h^2}{4} = \frac{3}{2}\frac{Q}{bh}. \qquad (9.19)$$

The area of the whole section is bh; denoting it by A, we obtain

$$\tau_{\max} = \frac{3}{2}\frac{Q}{A}. \qquad (9.20)$$

The quantity Q/A represents the average shearing stress; consequently, the maximum shearing stress for a rectangular section equals 1.5 times the average stress which would result if the shearing stresses were uniformly distributed across the depth of the section.

Formula (9.18) indicates that the shearing stresses vary across the section according to a parabolic law. For $y=h/2$, i.e., in the fibres most remote from the neutral axis, $\tau=0$. Figure 136*b* shows the shearing stress diagram for a rectangular section. Since y is quadratic in formula (9.18), the sign of y does not affect the magnitude of τ and fibres symmetrically removed from the neutral axis in a rectangular section have the same stresses.

The formula for determining shearing stresses in a beam of rectangular section subjected to bending was first derived by the prominent Russian engineer D. I. Jourawski in 1855.

67. Shearing Stresses in an I-Beam

Knowing the law of distribution of shearing stresses for a rectangular section, it is possible to draw stress diagrams for other sections made up of rectangles, such as an I-section (Fig. 137*a*).

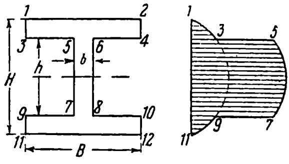
The shearing stresses in the horizontal flanges and the vertical web will vary according to a parabolic law. For points on lines *1-2* and *11-12* the stresses are zero. For points on lines *3-4* and *9-10* the shearing stresses can be determined from formula (9.17) by substituting the appropriate values. We then obtain

$$\tau_{3\text{-}4} = \frac{6Q}{B(H^3-\beta h^3)}\left(\frac{H^2}{4}-\frac{h^2}{4}\right) = \frac{3Q(H^2-h^2)}{2B(H^3-\beta h^3)}, \qquad (a)$$

where $\beta=(B-b)/B$. However, the stresses calculated by the formula (a) differ from the actual values, they are *conventional*; this follows from the fact that the uniform distribution of τ across the width of the section assumed in the derivation of formula (9.17)

does not exist in this case and, in particular, the stresses τ are zero in portions *3-5* and *6-4*. Therefore, as shown, the τ diagram in portions *1-3* and *9-11* is a conventionalized diagram (Fig. 137*b*).

In the transition to the vertical web the shearing stresses change sharply in magnitude since here the width changes abruptly from B to b. Therefore, for lines *5-6* and *7-8* belonging to the vertical



(a) (b)

Fig. 137

web the shearing stresses are determined from the expression (a) by substituting b for B. Thus,

$$\tau_{5-6}=\frac{3Q}{2b\,(H^3-\beta h^3)}\,(H^2-h^2). \tag{b}$$

The maximum stress occurs at the neutral axis. To determine it, we calculate the static moment of half the I-section with respect to the neutral axis and the moment of inertia of the whole section. The static moment is

$$S=B\frac{(H-h)}{2}\left(\frac{H}{2}-\frac{H-h}{4}\right)+b\frac{h}{2}\frac{h}{4}=\frac{B\,(H^2-h^2)}{8}+\frac{bh^2}{8}.$$

The moment of inertia of the I-section is

$$I=\frac{BH^3}{12}-\frac{(B-b)\,h^3}{12}.$$

Substituting the values of S and I in formula (9.17), we obtain

$$\tau_{\max}=\frac{Q\,\dfrac{B\,(H^2-h^2)+bh^2}{8}}{\dfrac{BH^3-(B-b)\,h^3}{12}\,b}=\frac{3Q\,[B\,(H^2-h^2)+bh^2]}{2\,[BH^3-(B-b)\,h^3]\,b}.$$

Figure 137*b* shows the shearing stress diagram for the I-section.

68. Verification of the Strength of a Beam on the Basis of Principal Stresses

The dimensions of the section of a beam must be chosen in such a manner that at none of the points the stresses exceed the allowable value. We saw above that both normal and shearing stresses are produced in a beam.

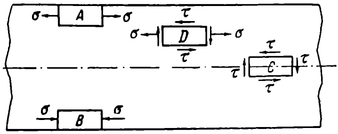


Fig. 138

In most cases the verification of the strength of beams is limited to the determination of the maximum normal stress from the formula

$$\sigma_{max} = \frac{M_{max}}{Z} \leqslant [\sigma]. \tag{9.16}$$

This strength condition applies to the elements most remote from the neutral layer at a section where the maximum bending moment occurs (Fig. 138).

An element (A) at one edge of the beam is in tension, an element (B) at the opposite edge is in compression.

The maximum shearing stress in the beam is checked by the formula

$$\tau_{max} = \frac{Q_{max}S}{Ib} \leqslant [\tau]. \tag{9.17}$$

This strength condition applies to an element located at the neutral layer at a section where the maximum shearing force occurs. This element (C) is in pure shear.

When moving from the edges of the beam to the neutral layer the normal stresses decrease and the shearing stresses increase. Therefore, if the strength of the beam is checked for only three elements, (A), (B) and (C), there is no assurance that we determine the maximum stresses in the beam. It may happen that the most stressed element is neither at the edges of the beam nor at the neutral layer. This occurs when the maximum bending moment and the maximum shearing force act at the same section of a beam, and the width of the section changes abruptly at the edges (for example, I-beam). We know how to determine the normal and shearing stresses for an element at any distance y from the

neutral layer if we have M and Q at a given section and the dimensions of the section itself.

Thus, we can easily determine the stresses σ and τ for the element (D) using formulas (9.3) and (9.17), respectively

$$\sigma=\frac{My}{I}, \qquad \tau=\frac{QS}{Ib}.$$

All the elements of the beam, except for elements located at the edges and at the neutral layer, are in a complex state of stress of the same type. The faces of an element which are perpendicular to the axis of the beam are acted on by normal (tensile or compressive) stresses; the same faces and those parallel to the neutral layer are acted on by shearing stresses; the front and back faces of the beam are free from stresses.

Knowing the stresses σ and τ for any one of such elements, we can find the principal stresses for this element and then evaluate the strength of the beam from its maximum equivalent stress on the basis of one or another strength theory.

When checking the overall strength of a beam, the principal stresses are determined from formulas (4.14) assuming $\sigma_y=0$, $\sigma_x=\sigma$, i. e., from the formulas

$$\sigma_{max}=\frac{1}{2}\sigma+\frac{1}{2}\sqrt{\sigma^2+4\tau^2},$$
$$\sigma_{min}=\frac{1}{2}\sigma-\frac{1}{2}\sqrt{\sigma^2+4\tau^2} \qquad (4.14)$$

and requiring that $\sigma_{max}\leqslant[\sigma]$.

Example 64. A simply supported I-beam of length $l=1$ m is bent by a force $P=16$ tons applied at mid-length (Fig. 139). Determine the principal stresses at points 1, 3 and 4 of the upper half of the most dangerous section of the beam. The dimensions of the section of the beam are indicated in centimetres.

Solution. An inspection of the bending moment and shearing force diagrams indicates that the most dangerous section of the beam is the middle section.

For this section the bending moment is

$$M_{max}=\frac{Pl}{4}=\frac{16{,}000\times100}{4}=400{,}000 \text{ kgf-cm}.$$

The shearing force is

$$Q=\frac{P}{2}=8{,}000 \text{ kgf}.$$

The moment of inertia of the section with respect to the neutral axis is

$$I=\frac{15\times40^3}{12}-\frac{(15-1)(40-2\times1.5)^3}{12}=20{,}900 \text{ cm}^4.$$

The normal stresses due to the bending moment and the shearing stresses due to the shearing force for the given points of the section are determined from the formulas

$$\sigma=\frac{My}{I},\qquad \tau=\frac{QS}{Ib}.$$

We first calculate the static moments

$$S_1=0,\quad S_3=15\times 1.5\left(\frac{40}{2}-\frac{1.5}{2}\right)=434\ \text{cm}^3,$$

$$S_4=S_3+1\left(\frac{40}{2}-1.5\right)\left(\frac{40}{2}-1.5\right)\frac{1}{2}=434+171=605\ \text{cm}^3.$$

Determine the normal stresses

$$\sigma_1=-\frac{400{,}000\times 20}{20{,}900}=-383\ \text{kgf/cm}^2,$$

$$\sigma_3=-\frac{400{,}000\left(\frac{40}{2}-1.5\right)}{20{,}900}=-354\ \text{kgf/cm}^2,\quad \sigma_4=0.$$

The normal stresses are negative since the fibres are compressed in the upper half of the section.

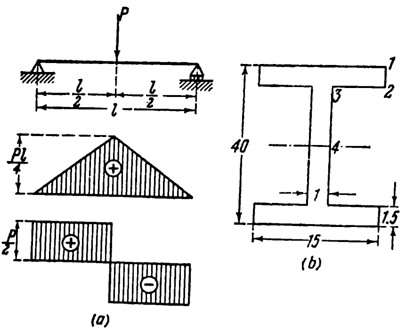


Fig. 139

Determine the shearing stresses

$$\tau_1=0\qquad \tau_3=\frac{8{,}000\times 434}{20{,}900\times 1}=166\ \text{kgf/cm}^2,$$

$$\tau_4=\frac{8{,}000\times 605}{20{,}900\times 1}=232\ \text{kgf/cm}^2.$$

The principal stresses are found by formulas (4.14)

$$\sigma_{1\,max} = 0, \quad \sigma_{1\,min} = -383 \text{ kgf/cm}^2;$$

$$\sigma_{3\,max} = -\frac{354}{2} + \frac{1}{2}\sqrt{354^2 + 4\times 166^2} = 66 \text{ kgf/cm}^2,$$

$$\sigma_{3\,min} = -\frac{354}{2} - \frac{1}{2}\sqrt{354^2 + 4\times 166^2} = -420 \text{ kgf/cm}^2,$$

$$\sigma_{4\,max} = 232 \text{ kgf/cm}^2, \quad \sigma_{4\,min} = -232 \text{ kgf/cm}^2.$$

From this example it is seen that the greatest principal stresses may occur in other than extreme fibres when the magnitude of the shearing force is so large that its effect is comparable with the effect of the bending moment.

69. Design of Beams Based on Allowable Loads, and Limit Design

In the above discussion the choice of the section of a beam was based on the method of allowable stresses. We assumed that failure occurred when the maximum stress at the dangerous section reached the yield point stress. The strength condition was expressed then as

$$\sigma_{max} = \frac{M_{max}}{Z} \leqslant [\sigma] = \frac{\sigma_y}{k},$$

where σ_y is the yield strength of the material and k is the factor of safety.

As was shown in the analysis of the torsion of shafts, when the stresses are not uniformly distributed over the section and the

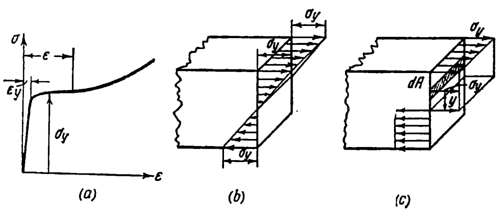


Fig. 140

material is ductile, the method of determining the dimensions of the section on the basis of the allowable loads gives a different result than the method of allowable stresses though the factor of safety remains the same. This is also true for the bending of a beam of ductile material whose tension test diagram is shown schemati-

cally in Fig. 140*a*. When the stress reaches the value σ_y in the extreme fibres at the most dangerous section of the beam, the stress diagram will be of the form shown in Fig. 140*b*. As the load continues to increase the maximum stress in the extreme fibres will not increase due to yielding of the material. The stresses in fibres nearer the neutral line will increase with increasing load.

At a certain value of the load when the stresses in the material reach the yield point stress throughout the depth of the section, i. e., the stress diagram takes the form of two rectangles (Fig. 140*c*), the load-carrying capacity of the beam will be exhausted. Determine the magnitude of the external bending moment M_{lim} corresponding to this limiting state of the beam.

The shaded elementary area dA is acted on by the force $\sigma_y\,dA$; the moment due to this force about the neutral axis is $\sigma_y y\,dA$. Consequently,

$$M_{\text{lim}}=\int\limits_A \sigma_y y\,dA$$

or because of the symmetry of the section about the neutral line

$$M_{\text{lim}}=2\sigma_y\int\limits_0^{h/2} by\,dy=\frac{\sigma_y bh^2}{4}$$

(σ_y could be put before the integral sign since it is constant throughout the section).

For a factor of safety k, the strength condition is

$$[M]\leqslant\frac{M_{\text{lim}}}{k}=\frac{\sigma_y bh^2}{4k}$$

or, introducing the allowable stress $[\sigma]=\sigma_y/k$, we obtain

$$[M]\leqslant\frac{[\sigma]\,bh^2}{4} \tag{9.21}$$

or

$$[\sigma]\geqslant\frac{4\,[M]}{bh^2}=\frac{M}{\frac{3}{2}Z}. \tag{9.22}$$

In the design based on the allowable stress, we had

$$[\sigma]\geqslant\frac{M}{Z}.$$

Consequently, in the case of a rectangular beam the required section modulus in the design based on the allowable load is 1.5 times smaller than in the design based on the allowable stress for the same factor of safety.

Let us find the ratio between the weights of beams of rectangular section designed by the two methods. Suppose that the width of the section of a beam designed on the basis of the allowable stress is b, the depth h, the cross-sectional area A, the section modulus Z. The same quantities for the section of a beam designed on the basis of the allowable load are denoted by b_1, h_1, A_1 and Z_1, respectively. The cross sections of the beams are assumed similar, i. e., $b_1=\alpha b$, $h_1=\alpha h$. On the basis of the result obtained above

$$\frac{Z}{Z_1}=\frac{bh^2}{b_1h_1^2}=\frac{bh^2}{\alpha^3 bh^2}=1.5 \text{ or } \alpha=\sqrt[3]{\frac{1}{1.5}}.$$

The ratio of the cross-sectional areas is found to be

$$\frac{A}{A_1}=\frac{bh}{\alpha^2 bh}=\frac{1}{\left(\sqrt[3]{\frac{1}{1.5}}\right)^2}=1.31.$$

Thus, the saving in the weight of the beam is 31 per cent in this case.

We shall also derive a formula for solving beam problems in compliance with the requirements of the *limit design* method laid down by the Structural Design Code (see Sec. 20, p. 73). By analogy with the previously developed formulas (I) and (II) used in designs for central tension, we write expressions for the bending moments, $M_{l.c.}$, taking account of the load-carrying capacity of the beam in bending, and M_d, the design moment

$$M_{l.c}=ZR_b^n km, \tag{a}$$

$$M_d=M_1n_1+M_2n_2. \tag{b}$$

Here, as before, k and m are the factors of material non-homogeneity and operating conditions, respectively; the correction due to natural non-homogeneity revealed by statistical methods requires that the factor k be less than 1; as regards the factor m, it takes account, as before, of stress concentration effects (if required by the operating conditions of the beam) and any other effects reducing the load-carrying capacity of the beam. In the formula (*a*), R_b^n and Z are, respectively, the normative strength of the material in bending and the section modulus for the cross section of the beam.

The effect of a dead (sustained) load is determined by the magnitude of the bending moment M_1 as calculated from the M diagram which is plotted by the rules already familiar to us; here it is also necessary to take into account a possible overloading due to the dead load by incorporating the factor $n_1>1$. A similar estimate of the effect of a live (short-time) load is made by means of the bending moment M_2 and the overloading factor $n_2>1$. The quantities M and n are included in the formula (b). Comparing the

values of $M_{l.c.}$ and M_d, we obtain the *design formula*

$$M_1 n_1 + M_2 n_2 \leqslant Z R_b^n km. \tag{c}$$

Suppose that it is required to design a simply supported beam of span $l = 4$ m carrying a dead load (uniformly distributed throughout the span) of intensity $q = 0.5$ ton/m and a live load P (concentrated at mid-span) of 10 tons. The design factors have the following values: $n_1 = 1.1$, $n_2 = 1.3$, $k = m = 0.9$. The normative strength of steel is $R_b^n = 2.5$ tons/cm². Determine the required number of I-beam according to GOST (USSR State Standard).

The bending moments are (at mid-span where they are maximum): $M_1 = ql^2/8 = 1$ ton-m, $M_2 = Pl/4 = 10$ tons-m. From the formula (c) we find the required section modulus

$$Z = \frac{1\times 1.1 + 10\times 1.3}{2{,}500\times 0.9^2}\,10^5 = 696 \text{ cm}^3.$$

Referring to a GOST table, we find a No. 36 I-beam with $Z = 743$ cm³.

The formula (c) can be reduced to the form corresponding to designs based on elastic action; to do this, we divide both sides of the formula (c) by n_2; denoting the left-hand side by σ, we have

$$\sigma = \frac{M_1 \frac{n_1}{n_2} + M_2}{Z} \leqslant R_b^n \frac{km}{n_2}; \tag{d}$$

the quantity appearing on the right-hand side of this expression may be regarded as the *normative allowable stress* in the way it is used in designs based on allowable stresses, i. e.,

$$[\sigma] = R_b^n \frac{km}{n_2}. \tag{e}$$

Referring to the example just considered, we obtain from this formula: $[\sigma] = 1{,}560$ kgf/cm².

In this manner the design based on the first limiting state is carried out in the new design method. The design based on the second limiting state requires the use of the formula (IV) derived in Sec. 20 (p. 74) according to which the actual deflection Δ of the beam is calculated by the methods of Sec. 72 and compared with the allowable deflection f specified by the code rules ($\Delta \leqslant f$).

The most essential features of the new design method are: a clear idea of safety with respect to a dead and a live load (separately), variation of data on the material and operating conditions.

70. Check Questions

How do normal stresses vary over the cross section of a beam under bending?

Define the section modulus.

What are the section moduli of a rectangle and a circle?
Write the design flexure formula.
Write the formula for shearing stresses in bending.
What are the shearing stresses in the extreme fibres of a beam under bending?
What are the maximum shearing stresses on the cross section of a rectangular beam under bending?
When is it advisable to choose a cross section of a beam that is not symmetrical about the neutral axis?
How can the strength of a beam be checked on the basis of principal stresses?
What is the difference between beam designs based on the allowable load and the allowable stress?
State the concepts of the limit design of structures.
What determines the load-carrying capacity of a beam?
What is the reason for using different overloading factors for a dead and a live load?
Analyse formula (9.1) and show that the formula for σ should be modified for materials having different moduli of elasticity in tension and compression (cast iron).

Chapter X

The elastic curve of a beam

71. The Elastic Curve of a Beam

The straight axis of a beam is deflected under the action of external loads. The deflected axis of a beam is called the *elastic curve*. The determination of the elastic curve of a beam is essential since it is often required that not only the stresses induced in the beam should not exceed the allowable stress but also the maximum deflection of the beam should not be greater than a certain predetermined value depending on the operating conditions of the beam. Besides, in the design of statically indeterminate beams, i.e., beams in which the number of reactions is greater than the number of the conditions of statics, the available equations are supplemented by equations derived from consideration of deformation.

In the analysis of stresses caused by bending (Sec. 63) we obtained formula (9.2)

$$\frac{1}{\rho}=\frac{M}{EI} \tag{9.2}$$

or

$$\rho=\frac{EI}{M}$$

which expresses *an important relation of the flexure theory*. In words this formula can be read as: *the radius of curvature* ρ *at any point of the elastic curve of a beam is directly proportional to the rigidity EI and inversely proportional to the bending moment*. If a beam is bent by a couple, the moment M is constant throughout its length and the radius of curvature is then also a constant, i.e., the elastic curve of the beam is a circular arc. In all other cases the elastic curve may have very different shapes. To obtain the equation of the elastic curve in rectangular coordinates, we agree to direct the x axis always along the axis of the beam to the right and the y axis vertically upward. The equation of the elastic curve must give a relation between the co-ordinates x and y of its points. With this relation available, it is possible to find the corresponding deflection y for any section of the beam at a distance x from the origin.

In mathematics, the following expression is derived for the radius of curvature at a point A with coordinates x, y (Fig. 141):

$$\rho = \pm \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}.$$

This exact expression for the radius of curvature may be replaced by a simpler approximate expression. The point is that the allowable deflections of beams subjected to bending are very small (about one thousandth of the length of the beam) and the elastic curve differs only slightly from a straight line. The quantity $\frac{dy}{dx}$ which represents $\tan\varphi$, i.e., the tangent of the angle that the line tangent to the elastic curve makes with the positive x axis, is so small that its squared value becomes negligible compared with unity to which it is added. Because of its smallness the expression $\left(\frac{dy}{dx}\right)^2$ can be neglected. The error introduced in the value of the radius of curvature does not exceed 0.5 per cent. In this case the radius of curvature of the elastic curve of the beam is represented in a simpler way

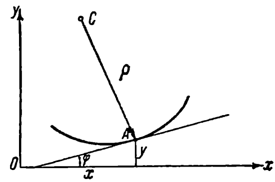


Fig. 141

$$\rho = \pm \frac{1}{\frac{d^2y}{dx^2}}. \tag{10.1}$$

Substituting this value of ρ in formula (9.2), we obtain the equation of the elastic curve in the form (differential form)

$$\pm EI\frac{d^2y}{dx^2} = M. \tag{10.2}$$

The two signs on the left-hand side of Eq. (10.2) are necessary in order to make the signs on the left-hand and right-hand sides always consistent. The choice of sign on the left-hand side is governed by the relative directions of the curvature of the elastic curve and the axis Oy; the sign on the right-hand side is that of the moment. If the curve is concave toward the positive y axis (Fig. 142), then $\rho > 0$ since $\frac{d^2y}{dx^2} > 0$, and conversely if the curve is concave toward the negative y axis (Fig. 143), then $\rho < 0$ since $\frac{d^2y}{dx^2} < 0$.

This, if we agree to choose the positive direction of the y axis upward, the sign of ρ $\left(\text{or } \frac{d^2y}{dx^2}\right)$ will coincide with the sign of the bending moment, as can easily be observed from examination of Figs. 142 and 143. From this it follows that, if the co-ordinate axes are chosen as stated above, the elastic curve equation may be written in the general form as

$$EI\frac{d^2y}{dx^2}=M. \tag{10.3}$$

To obtain the elastic curve equation in a form giving a direct relation between deflection y and abscissa x, it is necessary to integrate Eq. (10.3) twice. The first integration yields an equation relating

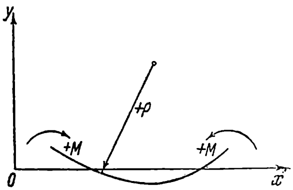


Fig. 142

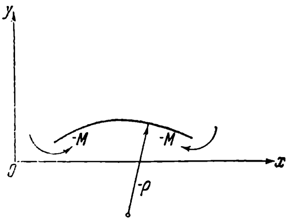


Fig. 143

the tangents of the angles $\left(\frac{dy}{dx}\right)$, that the lines tangent to elements of the elastic curve make with the x axis, to the abscissas of the elements. The second integration leads to the elastic curve equation in a form giving a direct relation between deflection y and abscissa x. After each integration a constant will result. Thus, we shall have two constants of integration for each portion of the beam after a double integration of its elastic curve equation.

When the number of portions of the beam is large, this method involves the solution of a system of equations with a large number of unknown constants. These constants are determined from the conditions of equality of deflections and angles of rotation at the boundaries of adjacent portions and from the behaviour of the beam at supports. However, adhering to certain rules and procedures of setting up and integrating the bending moment equations in separate portions, it is always possible to reduce the number of unknowns to two. This greatly simplifies the problem of finding the elastic curve of a beam having several portions.

To begin with, we agree to choose the origin of co-ordinates at the left end of a beam with x positive to the right and y positive

upward. In calculating moments we shall consider the part of the beam which contains the origin, i.e., we shall always determine the moment at a given section approaching it from the left.

We now proceed to the description of three relevant procedures which will be illustrated by examples.

The first procedure consists in integrating some expressions containing parentheses without opening them. For example, the integra-

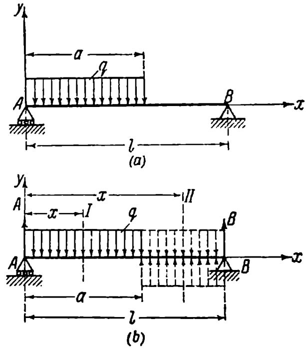


Fig. 144

tion of an expression of the form $P(x-a)$ is performed without opening the parentheses, i.e., according to the following formula:

$$\int P(x-a)^m dx = P\frac{(x-a)^{m+1}}{m+1}+C.$$

The integration by this formula differs from an integration where the parentheses are opened first only in the magnitude of an arbitrary constant.

The second procedure consists in the following. If a beam is subjected to a distributed load which does not reach the end of the beam, it should be extended up to the end and at the same time a load of the same intensity, equal magnitude but opposite in sign to the added load should be applied in order to leave the operating conditions of the beam unaltered.

If, for example, a beam is acted on by a uniformly distributed load of intensity q which does not reach the end, as shown in Fig. 144*a*, this load should be extended up to the end of the beam (Fig. 144*b*), at the same time applying a load equal but opposite to the added one. The two added loads are shown dashed in Fig. 144*b*.

The third procedure will be illustrated by an example. Let a beam be acted on by a concentrated moment m (Fig. 145) at a distance a from the left support. The bending moment in the second portion is, if the section is approached from the origin, as we agreed,

$$Ax - m.$$

Nothing will be changed if we write this moment as

$$Ax - m(x-a)^0,$$

i.e., we have introduced a factor $(x-a)^0$ equal to unity (a is the length of the beam from the origin to the section where the concentrated moment m is applied). The third procedure consists in

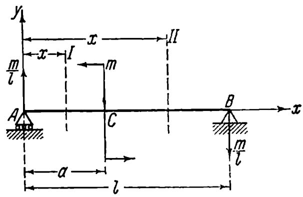


Fig. 145

multiplying the concentrated moment by a factor $(x-a)^0$ equal to unity.

In deriving the generalized equation of the elastic curve in the next section it will be shown that, if the foregoing procedures are followed up, there never will be more than two arbitrary constants of integration regardless of the number of portions of the beam.

72. Derivation of the Generalized Equation of the Elastic Curve

Let a beam be in equilibrium under the action of positive loads (producing positive bending moments) indicated in Fig. 146. The origin of co-ordinates is taken at point O, the x axis is chosen along the axis of the beam to the right, the y axis vertically upward.

Consider five portions of the beam.

Portion I: OA. There is no load in the first portion; consequently, the equations defining the elastic curve are

$$EI\frac{d^2y}{dx^2} = 0, \qquad EI\frac{dy}{dx} = C_1, \qquad EIy = C_1x + D_1.$$

Portion II: *AB*. Apply the third procedure

$$EI\frac{d^2y}{dx^2}=m(x-a)^0.$$

The integration of this equation is performed applying the first

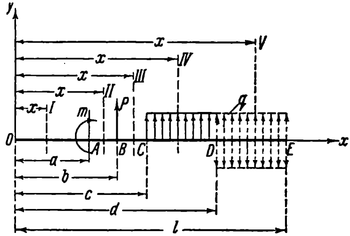


Fig. 146

procedure

$$EI\frac{dy}{dx}=m(x-a)+C_2,$$

$$EIy=m\frac{(x-a)^2}{2}+C_2x+D_2.$$

Portion III: *BC*.

$$EI\frac{d^2y}{dx^2}=m(x-a)^0+P(x-b),$$

$$EI\frac{dy}{dx}=m(x-a)+P\frac{(x-b)^2}{2}+C_3,$$

$$EIy=m\frac{(x-a)^2}{2}+P\frac{(x-b)^3}{6}+C_3x+D_3.$$

Portion IV: *CD*.

$$EI\frac{d^2y}{dx^2}=m(x-a)^0+P(x-b)+q\frac{(x-c)^2}{2},$$

$$EI\frac{dy}{dx}=m(x-a)+P\frac{(x-b)^2}{2}+q\frac{(x-c)^3}{6}+C_4,$$

$$EIy=m\frac{(x-a)^2}{2}+P\frac{(x-b)^3}{6}+q\frac{(x-c)^4}{24}+C_4x+D_4.$$

Portion V: *DE*. The distributed load does not extend to the fifth portion; therefore, to obtain equal constants of integration we add, according to the second procedure, a positive load and

a similar negative load to keep the operating conditions of the beam unchanged. The added loads are shown dashed in the drawing.

We have for the fifth portion

$$EI\frac{d^2y}{dx^2}=m(x-a)^0+P(x-b)+q\frac{(x-c)^2}{2}-q\frac{(x-d)^2}{2},$$

$$EI\frac{dy}{dx}=m(x-a)+P\frac{(x-b)^2}{2}+q\frac{(x-c)^3}{6}-q\frac{(x-d)^3}{6}+C_5,$$

$$EIy=m\frac{(x-a)^2}{2}+P\frac{(x-b)^3}{6}+q\frac{(x-c)^4}{24}-q\frac{(x-d)^4}{24}+C_5x+D_5.$$

The equality of constants ($C_1=C_2=\ldots=C$ and $D_1=D_2=\ldots$ $\ldots=D$) follows from comparison of the relevant equations after substituting the values of x corresponding to the boundary between two adjacent portions. Thus, in order to prove the equality $C_3=C_4$ we substitute $x=c$ in the equations for the slopes of tangents in the third and fourth portions. We obtain

$$m(c-a)+P\frac{(c-b)^2}{2}+C_3=m(c-a)+P\frac{(c-b)^2}{2}+q\frac{(c-b)^3}{6}+C_4$$

from which it follows that $C_3=C_4$. After proving the equality of all the constants C, the equality for the D's can easily be proved in exactly the same manner.

The physical meaning of the constants C and D becomes clear from consideration of the elastic curve in the first portion.

If the slope of the tangent to the elastic curve at the origin is denoted by α_0 and the deflection at the same section by f_0, we obtain from the equations for the slopes of tangents and deflections of the first portion putting $x=0$

$$EI\alpha_0=C,$$
$$EIf_0=D.$$

Consequently, the constant C represents the slope of the tangent at the origin multiplied by the flexural rigidity of the beam EI, and the constant D is the deflection at the origin multiplied by the same flexural rigidity EI.

Let us substitute the values of the constants C and D in the slope and deflection equations of the fifth portion as in the most general equations containing all the bending factors (couple, concentrated load and distributed load).

The equation for the slopes of tangents is then

$$EI\frac{dy}{dx}=EI\alpha_0+m(x-a)+P\frac{(x-b)^2}{2}+q\frac{(x-c)^3}{6}-q\frac{(x-d)^3}{6}.$$

The equation for deflections is

$$EIy=EIf_0+EI\alpha_0x+m\frac{(x-a)^2}{2}+P\frac{(x-b)^3}{6}+q\frac{(x-c)^4}{24}-q\frac{(x-d)^4}{24}.$$

When the loads acting on a beam are multiply repeated, these equations may be written in a more general form

$$EI\frac{dy}{dx}=EI\alpha_0+\Sigma m(x-a)+\Sigma P\frac{(x-b)^2}{2}+\Sigma q\frac{(x-c)^3}{6}-\Sigma q\frac{(x-d)^3}{6}, \quad (10.4)$$

$$EIy=EIf_0+EI\alpha_0 x+\Sigma m\frac{(x-a)^2}{2}+\Sigma P\frac{(x-b)^3}{6}+$$
$$+\Sigma q\frac{(x-c)^4}{24}-\Sigma q\frac{(x-d)^4}{24}. \quad (10.5)$$

Equations (10.4) and (10.5) are called the *generalized* (for the types of loading here considered) or *universal equations* of the elastic curve.

It should be noted that Fig. 146 shows positive directions of loads. If a load acts in the opposite direction, it is given a minus sign in Eqs. (10.4) and (10.5).

The direction of deflection is determined by its sign; when the sign is positive, the deflection is in the direction of the positive y axis, i. e., upward; when the sign is negative, the deflection is directed downward.

If a beam is fixed, the unknowns α_0 and f_0 vanish (the fixed end coincides with the origin) since the slope of the tangent and the deflection are zero at the fixed end.

If a beam is simply supported and possibly overhangs at one end, it remains to determine only one unknown α_0 since the deflection at the left support coinciding with the origin is zero. The unknown α_0 is determined in this case from the condition of zero deflection over the second, right, support.

Finally, in the third case when a simply supported beam has an origin which is chosen at a point other than a support, we have to determine two unknowns, α_0 and f_0. These unknowns are determined from the conditions of zero deflections over the supports. This variant is convenient when either α_0 or f_0 vanishes (see Example 65).

73. Special Cases of Determining Displacements of Beams from the Generalized Equation of the Elastic Curve

Below we shall consider several cases of determining displacements of beams most frequently encountered in practice.

Case 1. A beam fixed at one end is bent by a force P applied at the other end (Fig. 147). Determine the deflection under the force P.

Solution. The reaction at the fixed support is

$$A=P.$$

The moment at the support is

$$m=-Pl.$$

Since the slope of the tangent and the deflection are zero at the origin, i. e., $\alpha_0 = 0$ and $f_0 = 0$, the deflection at section B under the force will be written at once from Eq. (10.5). The equation will not contain the last two terms in this case as no distributed load is applied. The sign before the term containing the moment m is reversed since the support moment is negative. Thus, from Eq. (10.5) we have at $x = l$

$$EIy_B = A\frac{l^3}{6} - m\frac{l^2}{2}.$$

Substituting the values of A and m, we obtain

$$y_B = \frac{1}{EI}\left(P\frac{l^3}{6} - P\frac{l^3}{2}\right) = -\frac{Pl^3}{3EI}. \tag{10.6}$$

The minus sign indicates that the deflection is in the negative direction of the y axis, i. e., downward.

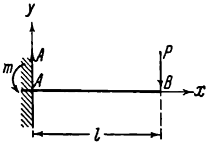


Fig. 147

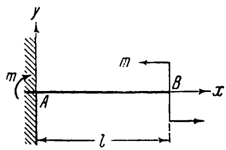


Fig. 148

Case 2. Determine the deflection of the free end of a cantilever beam (Fig. 148) bent by a moment m.

Solution. The moment at the fixed end is m. The deflection is written at once from Eq. (10.5) at $x = l$, since $\alpha_0 = 0$ and $f_0 = 0$

$$EIy_B = \frac{ml^2}{2}$$

or

$$y_B = \frac{ml^2}{2EI}. \tag{10.7}$$

Case 3. Determine the deflection of the free end B of the beam shown in Fig. 149.

Solution. The reaction at the fixed support is

$$A = ql.$$

The moment at the support is

$$m = -\frac{ql^2}{2}.$$

At the origin $\alpha_0 = 0$ and $f_0 = 0$.

The deflection at the free end of the beam, i.e., at $x=l$, is, from Eq. (10.5),

$$EIy_B = A\frac{l^3}{6} - m\frac{l^2}{2} - q\frac{l^4}{24}.$$

The last two terms are taken with the minus sign since the moment m and the distributed load q are negative.

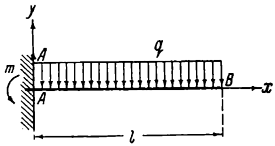


Fig. 149

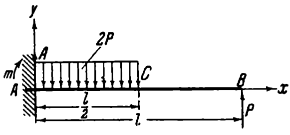


Fig. 150

Substituting the values of A and m, we obtain the deflection at section B

$$y_B = \frac{1}{EI}\left(q\frac{l^4}{6} - q\frac{l^4}{4} - q\frac{l^4}{24}\right) = -\frac{ql^4}{8EI}. \qquad (10.8)$$

Case 4. Determine the deflections at sections B and C of the beam shown in Fig. 150.

Solution. The reaction at the support is

$$A = P.$$

The moment at the support is

$$m = P\frac{l}{2}.$$

The intensity of distributed load is

$$q = \frac{4P}{l}.$$

At the origin $\alpha_0 = 0$ and $f_0 = 0$.

The deflection at section C, i.e., at $x = l/2$, is, from Eq. (10.5),

$$EIy_C = A\frac{\left(\frac{l}{2}\right)^3}{6} + m\frac{\left(\frac{l}{2}\right)^2}{2} - q\frac{\left(\frac{l}{2}\right)^4}{24} = A\frac{l^3}{48} + m\frac{l^2}{8} - q\frac{l^4}{384}.$$

The distributed load extends up to section C, therefore the last term of Eq. (10.5) vanishes; the minus sign is put before q since the distributed load is negative. Substituting the values of A, m and q, we obtain

$$y_C = \frac{1}{EI}\left(P\frac{l^3}{48} + P\frac{l^3}{16} - P\frac{l^3}{96}\right) = \frac{7Pl^3}{96EI}.$$

The deflection at section B, i.e., at $x=l$, is, from Eq. (10.5),

$$EIy_B = A\frac{l^3}{6} + m\frac{l^2}{2} - q\frac{l^4}{24} + q\frac{\left(\frac{l}{2}\right)^4}{24}.$$

Substituting the values of A, m and q, we find

$$y_B = \frac{1}{EI}\left(P\frac{l^3}{6} + P\frac{l^3}{4} - P\frac{l^3}{6} + P\frac{l^3}{96}\right)$$

or

$$y_B = \frac{25Pl^3}{96EI}.$$

Case 5. A simply supported beam is bent by a concentrated force P (Fig. 151). Determine the deflection under the force P and the maximum deflection of the beam.

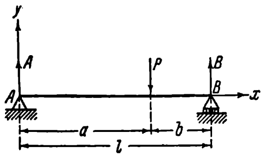


Fig. 151

Solution. The reactions at the supports are

$$A = P\frac{b}{l}, \qquad B = P\frac{a}{l}.$$

At the origin (point A) the deflection is zero, $f_0=0$, but $\alpha_0 \neq 0$. The slope α_0 at the origin is found from the condition of zero deflection over support B. From Eq. (10.5) we obtain at $x=l$

$$0 = EI\alpha_0 l + A\frac{l^3}{6} - P\frac{(l-a)^3}{6},$$

whence

$$EI\alpha_0 = P\frac{b^3}{6l} - A\frac{l^2}{6} = P\frac{b^3}{6l} - P\frac{bl^2}{6l}$$

or

$$\alpha_0 = -P\frac{b(l^2-b^2)}{6lEI}. \tag{10.9}$$

The deflection under the force P is determined from Eq. (10.5) putting $x=a$

$$EIy_P = EI\alpha_0 a + A\frac{a^3}{6}$$

or

$$y_P = -\frac{Pab(l^2-b^2)}{6lEI} + \frac{Pba^3}{6lEI} = -\frac{Pab(a^2+2ab+b^2-b^2-a^2)}{6lEI} = -\frac{Pa^2b^2}{3EIl}. \tag{10.10}$$

To determine the maximum deflection of the beam, it is necessary to know its location along the span. With this aim in view,

we use the condition that the tangent at the point of maximum deflection is parallel to the axis Ox, i.e., $\frac{dy}{dx}=0$. From the equation for the slopes of tangents (10.4) in the first portion, we have, assuming $a>b$,

$$0=-P\frac{b(l^2-b^2)}{6l}+A\frac{x^2}{2}=-P\frac{b(l^2-b^2)}{6l}+P\frac{b}{l}\frac{x^2}{2},$$

whence

$$x=\sqrt{\frac{l^2-b^2}{3}}.$$

The equation for deflections in the first portion is

$$EIy=EI\alpha_0 x+A\frac{x^3}{6}=-P\frac{b(l^2-b^2)}{6l}x+P\frac{b}{l}\frac{x^3}{6}.$$

Substituting the value of x found above, we obtain

$$EIy_{\max}=-P\frac{b(l^2-b^2)}{6l}\sqrt{\frac{l^2-b^2}{3}}+P\frac{b}{l}\frac{\left(\sqrt{\frac{l^2-b^2}{3}}\right)^3}{6}$$

or

$$y_{\max}=\frac{-Pb(l^2-b^2)\sqrt{3(l^2-b^2)}}{27EIl}. \tag{10.11}$$

In the particular case when the force P acts at the middle of the beam, i.e., when $a=b=l/2$, the angles of rotation of the sections at the supports are equal

$$\left.\begin{aligned}\alpha_0&=-\frac{Pl^2}{16EI},\\ \alpha&=\frac{Pl^2}{16EI}.\end{aligned}\right\} \tag{10.12}$$

The maximum deflection in this particular case is under the force

$$y_{\max}=-\frac{Pl^3}{48EI}. \tag{10.13}$$

Case 6. A simply supported beam is bent by a moment acting at mid-span (Fig. 152). Determine the slopes of the tangents to the elastic curve over the supports and the maximum deflection of the beam.

Solution. The reactions of the given beam have been determined on p. 194 and are expressed as

$$A=B=\frac{m}{l}.$$

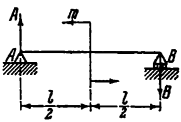


Fig. 152

The slope of the tangent to the elastic curve at the origin, i.e., over support A, is determined from the

condition of zero deflection over support B, i.e., at $x=l$. From Eq. (10.5) we have

$$EIy_B = EI\alpha_0 l + \frac{m}{l}\frac{l^3}{6} - \frac{m\left(\frac{l}{2}\right)^2}{2} = 0,$$

whence

$$EI\alpha_0 l = \frac{ml^2}{8} - \frac{ml^2}{6}$$

or

$$\alpha_0 = -\frac{ml}{24EI}.$$

The slope of the tangent to the elastic curve over support B is determined from Eq. (10.4), putting $x=l$,

$$EI\alpha_B = EI\alpha_0 + \frac{m}{l}\frac{l^2}{2} - m\frac{l}{2},$$

whence

$$\alpha_B = \alpha_0 = -\frac{ml}{24EI}. \tag{10.14}$$

The equation for deflections in the first portion of the beam is, according to Eq. (10.5),

$$EIy = EI\alpha_0 x + \frac{m}{l}\frac{x^3}{6} = -\frac{ml}{24}x + \frac{m}{l}\frac{x^3}{6}. \tag{a}$$

Determine the value of x at which the deflection in the first portion has a maximum value. To do this, we write the expression for $\frac{dy}{dx}$ and equate it to zero

$$\frac{dy}{dx} = \frac{1}{EI}\left(-\frac{ml}{24} + \frac{m}{l}\frac{x^2}{2}\right) = 0$$

or

$$-\frac{l}{24} + \frac{x^2}{2l} = 0,$$

whence

$$x = \sqrt{\frac{2l^2}{24}} = \frac{l}{2\sqrt{3}}.$$

The maximum deflection in the first portion is found by substituting the above value of x in the equation for deflections (a)

$$EIy_{\max} = -\frac{ml}{24}\frac{l}{2\sqrt{3}} + \frac{m}{l}\frac{l^3}{144\sqrt{3}},$$

whence

$$y_{max} = -\frac{ml^2}{EI\sqrt{3}}\left(\frac{1}{48}-\frac{1}{144}\right) = -\frac{ml^2}{72\sqrt{3}EI}. \tag{10.15}$$

Determine the deflection at the middle of the beam, i.e., at $x=l/2$. From the equation (a) we obtain

$$EIy_C = -\frac{ml}{24}\frac{l}{2}+\frac{m}{l}\frac{l^3}{48}=0,$$

i.e., the deflection is zero at mid-length, the elastic curve has a point of inflection here.

The deflections in the second portion are the same as in the first portion but opposite in sign (positive).

Case 7. A simply supported beam (Fig. 153) is subjected to a continuous load of intensity q. Determine the deflection at the middle of the beam.

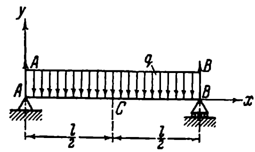


Fig. 153

Solution. The reactions at the supports are

$$A = B = q\frac{l}{2}.$$

Determine the slope α_0 of the elastic curve at the origin from the condition of zero deflection over support B. From Eq. (10.5) we have at $x=l$

$$0 = EI\alpha_0 l + A\frac{l^3}{6} - q\frac{l^4}{24},$$

whence

$$EI\alpha_0 = -A\frac{l^2}{6}+q\frac{l^3}{24} = -q\frac{l^3}{12}+q\frac{l^3}{24}$$

or

$$EI\alpha_0 = -q\frac{l^3}{24}. \tag{10.16}$$

The deflection at mid-length is found by substituting the value of $EI\alpha_0$ and $x=l/2$ in Eq. (10.5)

$$EIy_C = -q\frac{l^3}{24}\frac{l}{2}+A\frac{\left(\frac{l}{2}\right)^3}{6}-q\frac{\left(\frac{l}{2}\right)^4}{24}.$$

Substituting the value of the reaction A, we have

$$EIy_C = -q\frac{l^4}{48}+q\frac{l^4}{96}-q\frac{l^4}{384},$$

whence

$$y_C = -\frac{5ql^4}{384EI}. \tag{10.17}$$

Example 65. Determine the deflections at the ends and at the middle of the beam shown in Fig. 154; the load intensity is $q=2P/l$.

Solution. The reactions at the supports are $A=B=2P$. The origin is chosen, as usual, at the extreme left end of the beam. (If the origin is taken at point C, only the deflection f_{0C} need be found since the angle $\alpha_{0C}=0$ because of symmetry.)

At the origin $\alpha_0 \neq 0$ and $f_0 \neq 0$; consequently, before determining deflections, we must first find $EI\alpha_0$ and EIf_0 appearing in the

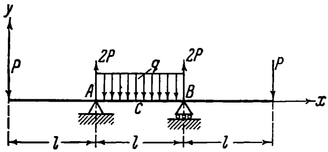


Fig. 154

equation for deflections (10.5). To determine these unknowns, we write two equations expressing the conditions of zero deflections over supports A and B, respectively,

$$0=EIf_0+EI\alpha_0 l-P\frac{l^3}{6},$$

$$0=EIf_0+EI\alpha_0 2l-P\frac{(2l)^3}{6}+2P\frac{l^3}{6}-\frac{2P}{l}\frac{l^4}{24}.$$

Solve this system of two equations

$$EIf_0+EI\alpha_0 l-P\frac{l^3}{6}=0, \quad \text{(a)}$$

$$EIf_0+2EI\alpha_0 l-\frac{13}{12}Pl^3=0. \quad \text{(b)}$$

Subtracting the first from the second equation, we obtain

$$EI\alpha_0 l-\frac{11}{12}Pl^3=0,$$

whence

$$EI\alpha_0=\frac{11}{12}Pl^2.$$

Substituting the value of $EI\alpha_0$ in the equation (a), we find

$$EIf_0=\frac{1}{6}Pl^3-\frac{11}{12}Pl^3=-\frac{3}{4}Pl^3.$$

From Eq. (10.5) we now determine the deflection at midpoint C, i. e., at $x=3l/2$

$$EIy_C=-\frac{3}{4}Pl^3+\frac{11}{12}Pl^2\,\frac{3}{2}l-P\frac{\left(\frac{3}{2}l\right)^3}{6}+$$
$$+2P\frac{\left(\frac{3}{2}l-l\right)^3}{6}-\frac{2P}{l}\,\frac{\left(\frac{3}{2}l-l\right)^4}{24}$$

or

$$y_C=\frac{19Pl^3}{192EI}.$$

74. Mohr's Method and Vereshchagin's Rule

We are often interested not in the entire elastic curve of a beam but only in the displacement at any one section. To determine the deflection or the angle of rotation of a beam it is then convenient to use Mohr's method which may also be applied to obtain the equation of the elastic curve.

Let the beam shown in Fig. 155a be bent by a load P and let it be required to determine the magnitude of the deflection y_C at section C. We take a similar beam (Fig. 155b) and load it at the same section by a force equal to unity (unit force). We now give this second beam additional deflections equal to those produced in the first beam by the load P. The additional potential energy U stored in the second beam is equal to the work done by the unit force during the unknown displacement y_C, i. e.,

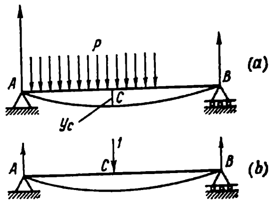


Fig. 155

$$U=1\times y_C \tag{a}$$

(the reactions at the supports due to the unit force do no work since there are no displacements at the supports)*. On the other hand, the additional potential energy of the second beam can be determined in a different way. Indeed, it is known from mechanics that the work done by a couple is equal to the moment of the couple times the angle of rotation. If we cut out from the second

* We do not take into account the energy required to produce the additional deformation itself; this energy is obviously equal to that stored in the first beam due to the action of the unit force.

beam an infinitesimal element of length dx, its potential energy is

$$dU = M_1 d\varphi,$$

where M_1 are the bending moments acting at the end sections of the isolated element due to the unit load, and $d\varphi$ is the angle of rotation of one end section of the element with respect to the other (Fig. 156). It should be noted that the angle of rotation of the end sections $d\varphi$ is, of course, caused not by the moments M_1 due to the unit load but by the moments M due to the load P acting on the first beam since the second beam was given the deflections of the first beam after it had been loaded by the unit force.

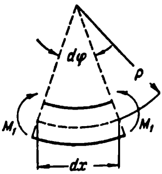


Fig. 156

From Fig. 156 it follows that $d\varphi = \frac{dx}{\rho}$ but since $\frac{1}{\rho} = \frac{M}{EI}$ [see formula (9.2)], $d\varphi = \frac{M\,dx}{EI}$ or

$$dU = M_1 \frac{M\,dx}{EI}.$$

The potential energy of the entire beam subjected to the unit force is

$$U = \int_0^l M_1 \frac{M\,dx}{EI}. \tag{b}$$

From comparison of the expressions (a) and (b) we obtain

$$y_C = \int_0^l M_1 \frac{M\,dx}{EI}. \tag{10.18}$$

If the beam is of uniform cross section, then

$$y_C = \frac{1}{EI} \int_0^l M_1 M\,dx. \tag{10.19}$$

Formulas (10.18) and (10.19) are known as Mohr's integrals. A semigraphical method of evaluating these integrals leads to Vereshchagin's rule. Consider this rule.

Take a portion of the beam of length $(l_2 - l_1)$ in which the bending moments due to the load acting on the beam are expressed in the general form by the equation $M = f(x)$ (Fig. 157a) and the bending moments due to the unit load vary according to a linear law, i.e., they are expressed by the equation $M_1 = ax + b$ (Fig. 157b)

Then the integral appearing in formula (10.19) may be expressed as

$$\int_{l_1}^{l_2} M_1 M\,dx = \int_{l_1}^{l_2} M(ax+b)\,dx = a\int_{l_1}^{l_2} Mx\,dx + b\int_{l_1}^{l_2} M\,dx.$$

The second of these integrals is the area under the M diagram in the portion of the beam under consideration; denote this area by A. The first integral represents the static moment of this area A with respect to a straight line perpendicular to the axis of the beam and passing through the origin; consequently, it is equal to Ax_c, where x_c is the abscissa of the centroid of the M diagram. Hence

$$\int_{l_1}^{l_2} M_1 M\,dx = aAx_c + bA =$$

$$= A(ax_c + b).$$

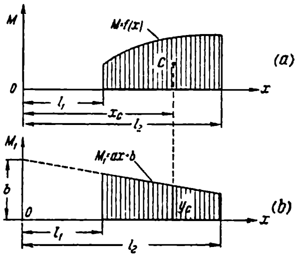


Fig. 157

The expression $ax_c + b$ represents the ordinate y_c of the diagram for moments M_1 due to the unit load at the section corresponding to the centroid of the M diagram. Thus,

$$\int_{l_1}^{l_2} M_1 M\,dx = Ay_c, \tag{10.20}$$

i. e., Vereshchagin's rule for evaluating Mohr's integral requires multiplying the area under the M diagram due to a given load by the ordinate of the M_1 diagram due to a unit load directly under the centroid of the M diagram.

To determine the displacement of a beam, formula (10.20) should be applied for all portions of the beam, then formula (10.19) for calculating the displacement will be expressed as

$$y = \frac{1}{EI}\Sigma Ay_c. \tag{10.21}$$

Example 66. Determine the deflection under a force P (Fig. 158*a*).

Solution. The area under the moment diagram due to the given load (Fig. 158*b*) is

$$A = -\frac{Pl^2}{2}.$$

The moment diagram due to a unit force applied at the section where the deflection is desired is shown in Fig. 158*d*. The ordi-

nate of this diagram under the centroid of the diagram due to the given load is $y_c = -\frac{2}{3}l$, by formula (10.21); the deflection under the force is

$$y_B = \frac{1}{EI}\Sigma A y_c = \frac{1}{EI}\left(-\frac{Pl^2}{2}\right)\left(-\frac{2}{3}l\right) = \frac{Pl^3}{3EI}.$$

The positive sign of y_B indicates that the direction of the deflection coincides with the direction of the unit force, i.e., the deflection is directed downward.

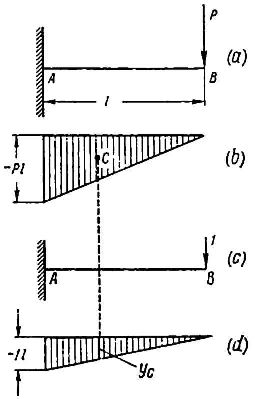


Fig. 158

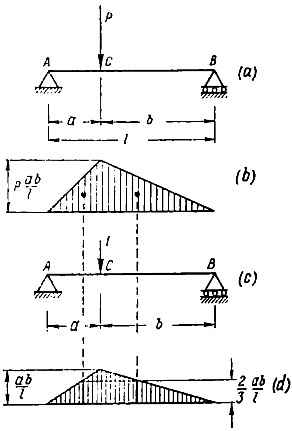


Fig. 159

Example 67. Determine the deflection under a force P (Fig. 159*a*). The moment diagram (Fig. 159*b*) is made up of two triangles whose areas are

$$A_1 = \frac{Pa^2b}{2l}, \qquad A_2 = \frac{Pab^2}{2l}.$$

The ordinates of the diagram due to a unit force under the centroids of the areas A_1 and A_2 are, respectively,

$$y_1 = \frac{2}{3}\frac{ab}{l}, \qquad y_2 = \frac{2}{3}\frac{ab}{l}.$$

The deflection under the force is, by formula (10.21),

$$y_C = \frac{1}{EI}(A_1y_1 + A_2y_2) = \frac{1}{EI}\frac{2}{3}\frac{ab}{l}\left(\frac{Pa^2b}{2l} + \frac{Pab^2}{2l}\right) = \frac{Pa^2b^2}{3EIl}.$$

Example 68. Determine the slope of the elastic curve of a beam (Fig. 160a) at section C where a moment is applied.

Solution. The slope of the elastic curve at any section is determined in exactly the same way as the deflection, the only difference being that a unit moment and not a unit force is applied at the section where the slope is desired. The moment diagram due to the given load is shown in Fig. 160b, it is made up of two triangles. The area under the diagram in the first portion is $A_1=ma^2/2l$, in the second portion $A_2=-mb^2/2l$.

The moment diagram due to a unit moment is shown in Fig. 160d; the ordinates of this diagram under the centroids of the areas A_1 and A_2 are, respectively,

$$y_1=\frac{2}{3}\frac{a}{l}, \qquad y_2=-\frac{2}{3}\frac{b}{l}.$$

The slope of the elastic curve of the beam at section C is

$$\alpha_C=\frac{1}{EI}(A_1y_1+A_2y_2)=\frac{1}{EI}\left[\frac{ma^2}{2l}\frac{2a}{3l}+\right.$$
$$\left.+\left(-\frac{mb^2}{2l}\right)\left(-\frac{2}{3}\frac{b}{l}\right)\right]=\frac{m(a^2+b^2-ab)}{3EIl}.$$

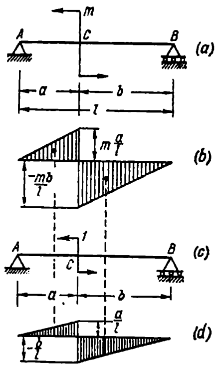


Fig. 160

Example 69. Determine the deflections at the ends and at the middle of a beam (Fig. 161a).

Solution. The moment diagram due to the given loads is shown in Fig. 161b. To determine the deflection at the end of the beam, we apply a unit force (Fig. 161c) and plot a moment diagram (Fig. 161d). The moment diagram due to the given loads has three portions; the areas under the diagrams in these portions are

$$A_1=-\frac{Pa^2}{2}, \qquad A_2=-2Pa^2, \qquad A_3=-\frac{Pa^2}{2}.$$

The ordinates of the moment diagrams due to the unit load under the centroids of the areas A_1, A_2 and A_3 are, respectively,

$$y_1=-\frac{2}{3}a, \qquad y_2=-\frac{a}{2}, \qquad y_3=0.$$

The deflection at the end of the beam is

$$y_C=\frac{1}{EI}(A_1y_1+A_2y_2+A_3y_3)=$$
$$=\frac{1}{EI}\left[\left(-\frac{Pa^2}{2}\right)\left(-\frac{2}{3}a\right)+(-2Pa^2)\left(-\frac{a}{2}\right)\right]=\frac{4}{3}\frac{Pa^3}{EI}.$$

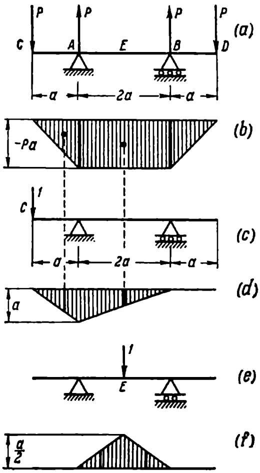


Fig. 161

Let us find the deflection at the middle section of the beam; we apply a unit force at this section (Fig. 161*e*) and plot a moment diagram (Fig. 161*f*). The ordinates of the moment diagram due to the unit force are zero in the overhangs while in the centre span this diagram is bilinear, therefore the area under the moment diagram due to the given load in the centre span should be divided into two parts; the areas under the diagram in the two portions are equal, each being $A = -Pa^2$. The ordinates of the diagram due to the unit load under the centroids of these areas are $y_c = a/4$. Therefore, the deflection at the middle of the beam is

$$y_E = \frac{2}{EI}(Ay_c) = \frac{2}{EI}(-Pa^2)\frac{1}{4}a = -\frac{Pa^3}{2EI}.$$

The minus sign indicates that the deflection is directed oppositely to the unit force, i.e., upward and not downward.

75. Beams of Uniform Resistance to Bending

So far we have considered prismatic beams in which the cross section remains uniform throughout the length of a beam. The cross-sectional dimensions of such beams are determined from the maximum bending moment M_{max} at the dangerous section. The stress on the dangerous section must not exceed the allowable stress $[\sigma]$

$$Z \geqslant \frac{M_{max}}{[\sigma]}. \tag{9.16}$$

It is clear that the stresses on all other sections of a prismatic beam will be smaller than the allowable value and only in pure bending are the stresses the same on all sections of a prismatic beam. In the latter case all the sections of a beam are equally dangerous. Thus, when a beam of uniform section is subjected to bending, save in the case of pure bending, all sections of the beam, except for the dangerous one, have an excessive reserve of strength which is indicative of irrational utilization of the material.

The most rational shape of a beam for a given loading is the one for which the stresses on all of its sections are equal to the allowable stress. Beams having a shape satisfying this condition are called *beams of uniform resistance to bending.*

If the bending moment at an arbitrary section of a beam of uniform resistance is denoted by M_x and the section modulus by

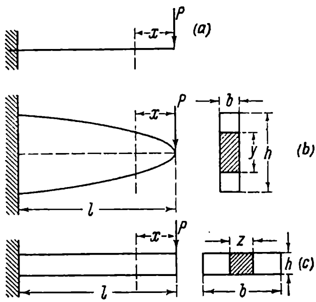


Fig. 162

Z_x, the following condition must be fulfilled

$$\frac{M_x}{Z_x}=\frac{M_{\max}}{Z}=\sigma=\text{constant}.$$

Hence

$$\frac{Z_x}{Z}=\frac{M_x}{M_{\max}}. \tag{10.22}$$

Consequently, in beams of uniform resistance to bending, the section moduli must be directly proportional to the respective bending moments.

The saving of material achieved in beams of uniform resistance does not always repay the more complex fabrication and therefore in practice the use of such beams is limited.

As an illustration to the theory of beams of uniform resistance let us consider the following example. A beam of rectangular section is fixed at one end and subjected to a concentrated force P at the other (Fig. 162*a*). How must the depth of the beam vary if the width is kept constant and how must the width of the beam vary if the depth is kept constant to provide uniform resistance to bending throughout the length of the beam? Determine tne deflection at the free end of a beam of uniform resistance and constant depth.

Solution. (1) Width is constant. The bending moment at an arbitrary section a distance x from the free end of the beam is

$$M_x = -Px.$$

The corresponding section modulus is (Fig. 162*b*)

$$Z_x = \frac{by^2}{6},$$

where b is the constant width of the beam and y the variable depth of the beam.

The bending moment on the section at the wall is

$$M_{max} = -Pl.$$

If the depth of the cross section of the beam at the wall is denoted by h, the corresponding section modulus is

$$Z = \frac{bh^2}{6}.$$

On the basis of Eq. (10.22) we obtain

$$\frac{\frac{by^2}{6}}{\frac{bh^2}{6}} = \frac{-Px}{-Pl},$$

whence

$$y^2 = \frac{h^2}{l} x. \tag{10.23}$$

The variation in the depth of a beam of uniform resistance to bending and constant width is shown in Fig. 162*b*.

Using the allowable stress, we can determine h from the equation

$$[\sigma] = \frac{M_{max}}{\frac{bh^2}{6}}$$

and hence the depth y at any section of the beam from Eq. (10.23).

The volume of a beam of uniform resistance is two-thirds of the volume of a beam of uniform section bh, i.e., saving in material is 33 per cent.

(2) Depth is constant. Denote the constant depth of a beam by h, the variable width by z and the width of the cross section of the beam at the wall by b. On the basis of Eq. (10.22) we have

$$\frac{Z_x}{Z} = \frac{\frac{zh^2}{6}}{\frac{bh^2}{6}} = \frac{-Px}{-Pl},$$

whence

$$z=\frac{b}{l}x. \tag{10.24}$$

Consequently, in this case the width of the beam varies according to a linear law. The shape of such a beam can easily be realized. It is shown in Fig. 162c. The use of such a beam provides a saving of material of 50 per cent as against a prismatic beam of section bh. Actually the saving will be somewhat smaller since the free end of the beam is made of constant width over a short length (Fig. 163); otherwise the shearing force at the end of the beam would cause too high shearing stresses. For this reason the free end is also strengthened in beams of uniform resistance to bending and constant width.

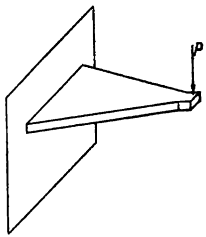


Fig. 163

(3) The deflection of the free end of a beam of uniform resistance and constant depth is determined as follows. On the basis of formula (9.2) of Sec. 63 we can write

$$\frac{EI_x}{\rho}=-Px,$$

where ρ denotes the radius of curvature of the elastic curve of the beam at an arbitrary section and I_x the moment of inertia of this section. In our case the moment of inertia is, taking into account Eq. (10.24),

$$I_x=\frac{\frac{b}{l}xh^3}{12}.$$

Consequently,

$$\frac{Ebxh^3}{12l\rho}=-Px$$

or

$$\rho=-\frac{Ebh^3}{12Pl}=-\frac{EI}{Pl},$$

where I denotes the moment of inertia of the section at the wall equal to $bh^3/12$. The right-hand side of the above expression is constant; hence, the radius of curvature ρ of the elastic curve of the beam is the same at all sections, i.e., the beam is bent to a circular arc (Fig. 164). From the right triangle OAB we have

$$\overline{OB}^2=\overline{OA}^2+\overline{AB}^2.$$

Since the deflection f is small compared with the radius of curvature ρ (Figure 164 is not drawn to scale), one half the chord, i.e., the line segment $\overline{AB}$ can be taken equal to the beam length l without much error. Consequently, we have

$$\rho^2=(\rho-f)^2+l^2,$$

whence

$$\rho^2=\rho^2-2\rho f+f^2+l^2.$$

Neglecting f^2 as a small quantity compared with the other quantities in this expression, we obtain

$$2\rho f=l^2,$$

whence

$$f=\frac{l^2}{2\rho}.$$

Substituting the radius of curvature $\rho=-\frac{EI}{Pl}$, we find

$$f=-\frac{Pl^3}{2EI}. \tag{10.25}$$

Comparing the above value of the deflection with that for a beam of uniform section

$$f=-\frac{Pl^3}{3EI}, \tag{10.6}$$

we conclude that the deflection of the beam of uniform resistance is 1.5 times that of the beam of uniform section.

This ability of a beam of uniform resistance to deform to a much greater extent than a beam of uniform section under the same loads and with the same allowable stresses accounts for the use of beams of uniform resistance in cases where it is necessary to alleviate the action of an impact load.

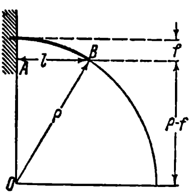


Fig. 164

Leaf springs widely used in various types of transport facilities (automobiles, railway carriages, etc.) are beams of uniform resistance. We shall illustrate the design of such springs by a numerical example.

Example 70. Determine the dimensions b and h of a steel leaf spring (Fig. 165a) of length $l=40$ cm which must deflect not less than 5 cm under a load $P=500$ kgf applied at the free end. The allowable stress in bending is $[\sigma]=5{,}000$ kgf/cm², $E=2.1\times10^6$ kgf/cm².

Solution. The maximum bending moment which occurs on the section at the wall is

$$M_{max}=Pl.$$

From the strength equation (9.16) we obtain

$$Z=\frac{bh^2}{6}\geqslant\frac{Pl}{[\sigma]},$$

whence

$$b\geqslant\frac{6Pl}{h^2[\sigma]}=\frac{6\times500\times40}{h^2 5,000}=\frac{24}{h^2}. \qquad \text{(a)}$$

The second equation for determining the unknown dimensions b and h is the deformation equation. On the basis of formula (10.25)

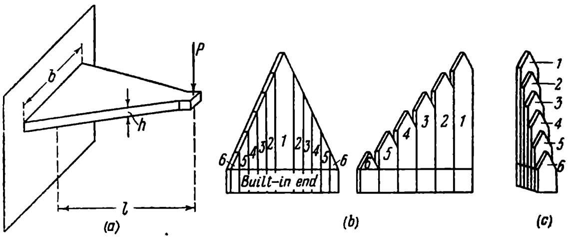


Fig. 165

we have

$$5=\frac{500\times40^3}{2\times2.1\times10^6\frac{bh^3}{12}}.$$

The minus sign before the right-hand side of the formula is omitted since in this case we are interested in the absolute value of the deflection and not in its direction.

We solve the last equation for b

$$b=\frac{500\times64,000\times12}{2\times2.1\times10^6\times5h^3}=\frac{18.3}{h^3}. \qquad \text{(b)}$$

From the expressions (a) and (b) we obtain

$$\frac{24}{h^2}=\frac{18.3}{h^3},$$

whence

$$h=\frac{18.3}{24}=0.763\text{ cm}.$$

Then from (a) we obtain

$$b=\frac{24}{0.763^2}=41.3\text{ cm}.$$

Round off the dimensions obtained and take

$$h=0.75\text{ cm},\qquad b=42\text{ cm}.$$

Check the stress and deflection provided by the chosen dimensions

$$\sigma=\frac{Pl}{\frac{bh^2}{6}}=\frac{500\times 40\times 6}{42\times 0.75^2}=5{,}100\ \text{kgf/cm}^2,$$

$$f=-\frac{Pl^3}{2E\frac{bh^3}{12}}=-\frac{500\times 40^3\times 12}{2\times 2.1\times 10\times 42\times 0.75^3}=-5.17\ \text{cm}.$$

The resulting stress is only 2 per cent higher than the allowable value; the deflection is somewhat larger than 5 cm, therefore, the assumed dimensions may be considered to satisfy the conditions stated in the problem.

In practice leaf springs are made in a somewhat different form. The considerable width of a spring where it is supported requires much space and this is often inconvenient. If a spring is cut into separate longitudinal strips of equal width (as shown in Fig. 165*b*) and the strips are placed on one another (as shown in Fig. 165*c*), the spring thus obtained will act in the same way as the whole one, neglecting the friction between the strips, while saving of space will be considerable. Of course, a leaf spring is not cut into separate strips but these strips are made from a fabricated long narrow steel strip and then arranged as stated above. In our example the spring may be made up, say, of six separate strips, then the width of each strip must be equal to

$$42:6=7\ \text{cm}.$$

76. Check Questions

What is the elastic curve of the beam?

What is the relation between the radius of curvature ρ, the bending moment M and the flexural rigidity EI?

What is the equation of the elastic curve in differential form? How can the equation of the elastic curve giving a direct relation between deflection y and abscissa x be derived from the equation of the elastic curve in differential form?

Write the generalized equation of the elastic curve.

What is the deflection of a cantilever beam bent by a force at its free end?

What is the deflection of a simply supported beam bent by a force at mid-length?

What is a beam of uniform resistance to bending?

Explain Mohr's and Vereshchagin's rules.

In what diagram is the ordinate not to be taken when using Vereshchagin's rule?

Check the deflection at mid-length (Example 69) taking the area from the diagram (d) (Fig. 161).

Chapter XI

Statically indeterminate beams

77. Concept of Statically Indeterminate Beams

It will be recalled that a statically indeterminate beam is a beam in which the total number of unknown reactions is greater than the number of available equations of statics expressing the conditions of equilibrium of the beam.

The so-called redundant unknown reactions impose additional deformation conditions on a beam. These conditions expressed mathematically provide the lacking number of equations for determining reactions; the determination of each redundant reaction requires an additional equation.

We have already encountered statically indeterminate problems in studying tension, compression and torsion. Their solution always called for the consideration of deformations. The determination of reactions of a statically indeterminate beam is also possible only by consideration of deformations. Thus, it may be said that in solving any statically indeterminate problem the redundant unknowns can be found by adding to the equations of statics the lacking number of equations obtained by consideration of deformations.

These additional deformation equations can be set up in a number of ways. One of the simple ways consists in applying the principle of superposition discussed in Sec. 20.

In studying statically indeterminate beams we shall limit ourselves to cases where the number of redundant unknowns is not great.

78. A Beam Fixed at One End and Simply Supported at the Other

The fixed end of a beam involves a force and a couple, the supported end a force. Thus, a beam with one end fixed and the other supported has three unknowns. In the case of forces acting in the same plane and perpendicular to the axis of the beam there are only two equations of equilibrium available for determining the reactions. Consequently, the beam under consideration has one redundant unknown.

The redundant unknown in this beam is taken as the reaction arising at the supported end.

By way of example we determine the maximum bending moment and the maximum deflection of the beam shown in Fig. 166*a*, which is subjected to a uniformly distributed load of intensity q.

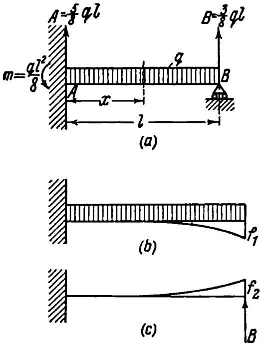


Fig. 166.

Determine first the redundant unknown reaction at support B. Removing support B, we obtain a statically determinate beam (Fig. 166*b*); the deflection of the free end of this type of beam was determined on p. 236; it is equal to

$$f_1 = -\frac{1}{8}\frac{ql^4}{EI}.$$

Actually the end of the beam is supported and its deflection is zero. Consequently, according to the principle of superposition the reaction at support B must be of such magnitude that the deflection determined above would be liquidated, i. e., this reaction, if it were acting separately, must produce a deflection f_2 equal in magnitude to f_1 but opposite in direction. This condition determines the reaction arising at support B.

The deflection due to the force B applied at the end of the beam (Fig. 166*c*) is

$$f_2 = \frac{1}{3}\frac{Bl^3}{EI}.$$

The sum of the deflections f_1 and f_2 must be zero, i. e.,

$$-\frac{1}{8}\frac{ql^4}{EI} + \frac{1}{3}\frac{Bl^3}{EI} = 0.$$

Hence

$$B = \frac{3}{8}ql.$$

After determining the reaction B we find, from the equilibrium conditions, the remaining unknowns, i. e., the reaction A at the fixed end and the moment m, as for a statically determinate beam (Fig. 166*a*).

The reaction is

$$A = ql - B = ql - \frac{3}{8}ql = \frac{5}{8}ql.$$

The moment m at the fixed end is determined by taking the sum of moments about A

$$-m+ql\frac{l}{2}-\frac{3}{8}qll=0,$$

whence

$$m=q\frac{l^2}{2}-\frac{3}{8}ql^2=\frac{ql^2}{8}. \qquad (11.1)$$

The moment equation for any section of the beam a distance x from the fixed end A is

$$M=Ax-m-qx\frac{x}{2}=\frac{5}{8}qlx-\frac{ql^2}{8}-\frac{qx^2}{2}.$$

To determine the value of x for which the bending moment is a maximum, we set the derivative $\frac{dM}{dx}$ equal to zero

$$\frac{dM}{dx}=\frac{5}{8}ql-qx=0,$$

whence

$$x=\frac{5}{8}l.$$

Consequently, the maximum moment is

$$M_{\max}=\frac{5}{8}ql\frac{5}{8}l-\frac{ql^2}{8}-\frac{q\left(\frac{5}{8}l\right)^2}{2}=\frac{9}{128}ql^2.$$

Comparing this moment with the moment at the fixed end, we see that the latter is larger than the maximum moment in the span of the beam, therefore the design moment is the moment at the fixed end.

The design equation for this beam is

$$[\sigma]\geqslant\frac{ql^2}{8Z}.$$

Determine now the maximum deflection of the beam.

The deflection equation is

$$EIy=-m\frac{x^2}{2}+A\frac{x^3}{6}-q\frac{x^4}{24}=-\frac{ql^2}{8}\frac{x^2}{2}+\frac{5}{8}ql\frac{x^3}{6}-\frac{qx^4}{24}.$$

The location of the maximum deflection is determined by setting the derivative $\frac{dy}{dx}$ equal to zero

$$\frac{dy}{dx}EI=-\frac{ql^2x}{8}+\frac{5}{16}qlx^2-\frac{qx^3}{6}=0,$$

$$x\left(\frac{x^2}{3}-\frac{5}{8}lx+\frac{l^2}{4}\right)=0.$$

The maximum deflection cannot occur at the fixed end since it is equal to zero there; consequently,

$$\frac{x^2}{3}-\frac{5}{8}lx+\frac{l^2}{4}=0$$

or

$$x^2-\frac{15}{8}lx+\frac{3}{4}l^2=0,$$

whence

$$x=\frac{15}{16}l\pm\sqrt{\frac{225}{256}l^2-\frac{3}{4}l^2}=\frac{15}{16}l\pm\frac{\sqrt{33}}{16}l=\frac{15\pm5.74}{16}l.$$

The value of x cannot be larger than l, therefore the minus sign is taken before the radical

$$x=\frac{15}{16}l-\frac{5.74}{16}l=\frac{9.26}{16}l=0.579l.$$

After determining the position of the section at which the maximum deflection occurs we find the magnitude of the deflection. To do this we substitute $x=0.579l$ in the deflection equation

$$EIy_{\max}=-\frac{ql^2}{8}\frac{(0.579l)^2}{2}+\frac{5}{8}ql\frac{(0.579l)^3}{6}-\frac{q(0.579l)^4}{24},$$

whence

$$y_{\max}=\frac{ql^4}{EI}(-0.02093+0.0202-0.00467)=-\frac{0.0054ql^4}{EI}=-\frac{ql^4}{185EI}.$$

Example 71. A beam AB built into a wall at one end and simply supported at the other is bent by a concentrated force P (Fig. 167*a*). Draw a moment and a shearing force diagram and determine the deflection under the force P.

Solution. The redundant unknown is taken as the reaction arising at support B. Removing support B, we obtain a statically determinate beam (Fig. 167*b*) the right end of which is deflected under the load P. To determine this deflection we must first find the reactions at the built-in end (Fig. 167*b*). From the equations of statics we have

$$A_1=P,\qquad m_1=Pa.$$

The senses of the reaction and the moment at the built-in end are indicated in Fig. 167*b*.

The deflection equation for the right-hand portion of the beam is, according to Eq. (10.5),

$$y=\frac{1}{EI}\left[-m_1\frac{x^2}{2}+A_1\frac{x^3}{6}-P\frac{(x-a)^3}{6}\right]$$

Substituting the values of A_1 and m_1 and the length of the beam l for x, we find the deflection of the right-hand end of the beam

$$f_1=\frac{1}{EI}\left[-Pa\frac{l^2}{2}+P\frac{l^3}{6}-P\frac{(l-a)^3}{6}\right].$$

Actually the end B of the beam is supported and its deflection is zero. Consequently, the reaction at support B must be of such

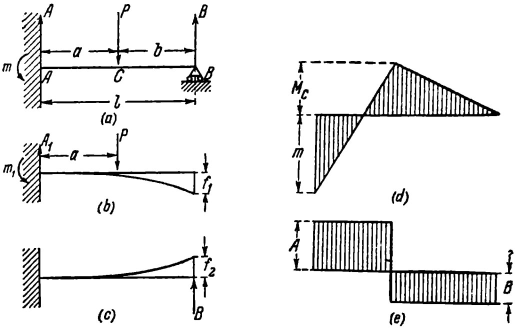


Fig. 167

magnitude that the deflection produced by it is equal in magnitude and opposite in direction to the deflection due to the force P.

The deflection of the right-hand end of the beam (Fig. 167c) due to the force B (p. 235) is

$$f_2=\frac{Bl^3}{3EI}.$$

From the condition

$$f_1+f_2=0$$

we determine the reaction B

$$\frac{1}{EI}\left[-Pa\frac{l^2}{2}+P\frac{l^3}{6}-P\frac{(l-a)^3}{6}\right]+\frac{Bl^3}{3EI}=0,$$

whence

$$B=P\left[\frac{3a}{2l}-\frac{1}{2}+\frac{(l-a)^3}{2l^3}\right]=P\frac{a^2(3l-a)}{2l^3}.$$

After determining the reaction B by consideration of the deformation of the beam we find, from the equilibrium conditions, the remaining unknowns, i. e., the reaction A and the moment at the

built-in end for the given beam (Fig. 167a). The reaction is

$$A = P\left[1 - \frac{a^2(3l-a)}{2l^3}\right].$$

The moment at the built-in end is

$$m = Pa - \frac{Pa^2(3l-a)}{2l^2} = \frac{Pa(2l^2-3al+a^2)}{2l^2} = \frac{Pab(l+b)}{2l^2}. \qquad (11.2)$$

To plot a moment diagram it is necessary to determine the moment at the section where the force P is applied. The moment at this section is

$$M_C = Bb = P\frac{a^2b(3l-a)}{2l^3}.$$

The moment and shearing force diagrams are shown in Fig. 167d and e.

To determine the deflection under the force P we write the deflection equation for the first portion of the beam

$$EIy = -m\frac{x^2}{2} + A\frac{x^3}{6}.$$

Substituting the values of m, A and $x = a$, we obtain

$$y_C = \frac{1}{EI}\left\{-P\frac{ab(l+b)}{2l^2}\frac{a^2}{2} + P\left[1 - \frac{a^2(3l-a)}{2l^3}\right]\frac{a^3}{6}\right\}.$$

In the particular case when the force P is applied at mid-length, we have

$$a = b = \frac{l}{2},$$

$$A = P\left[1 - \frac{l^2\left(3l - \frac{l}{2}\right)}{4\times 2l^3}\right] = \frac{11}{16}P,$$

$$B = \frac{5}{16}P,$$

$$m = \frac{5}{16}Pl - P\frac{l}{2} = -\frac{3}{16}Pl,$$

$$M_c = \frac{5}{16}P\frac{l}{2} = \frac{5}{32}Pl.$$

The design moment in this case is the moment at the built-in end. The deflection under the force is

$$y_C = \frac{1}{EI}\left[-\frac{3}{16}Pl\frac{\left(\frac{l}{2}\right)^2}{2} + \frac{11}{16}P\frac{\left(\frac{l}{2}\right)^3}{6}\right] = -\frac{7Pl^3}{768EI}.$$

79. A Beam with Both Ends Fixed

Each support of a beam fixed at both ends involves, in general, three reaction components, viz. a vertical force component, a horizontal force component and a couple. In the case of bending loads perpendicular to the axis of the beam the horizontal force component can be neglected since the stress produced by it in ordinary beams

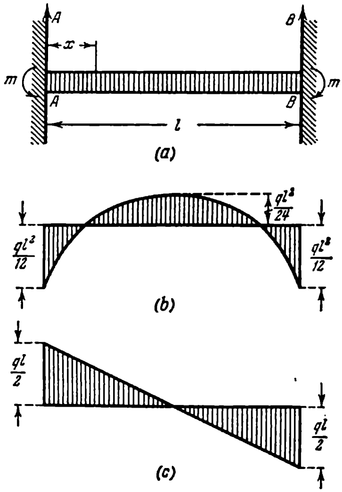


Fig. 168

is small compared with bending stresses. Thus, for a beam fixed at both ends and loaded with forces perpendicular to the axis there remain four reaction components of which two are statically indeterminate. Their determination requires four equations. Two equations are supplied by statics, the other two should be derived from additional deformation conditions. These may be the conditions of zero angles of rotation or zero deflections of the ends of the beam.

Consider an example. A beam AB is fixed at both ends and subjected to a uniformly distributed load of intensity q (Fig. 168a).

It is required to draw a moment and a shearing force diagram and determine the maximum deflection.

Because of the symmetry of the load the reactions at the supports are

$$A = B = \frac{ql}{2}.$$

For the same reason the moments m at the fixed ends are also equal.

Write the deformation equation. Since the slopes of the tangents to the elastic curve at the ends of the beam are zero, we obtain from Eq. (10.4)

$$0 = -ml + \frac{ql}{2}\frac{l^2}{2} + \frac{ql^3}{6},$$

whence

$$m = \frac{ql^2}{12}. \tag{11.3}$$

After determining the moments at the supports we proceed to the construction of a bending moment diagram. The beam under consideration has one portion. Write the moment equation for it

$$M = \frac{ql}{2}x - \frac{ql^2}{12} - q\frac{x^2}{2}.$$

The maximum moment in the portion is found by setting the derivative $\frac{dM}{dx}$ equal to zero. From this condition it follows that the maximum moment occurs at mid-length and is equal to

$$M_{\max} = \frac{ql}{2}\frac{l}{2} - \frac{ql^2}{12} - q\frac{\left(\frac{l}{2}\right)^2}{2} = \frac{ql^2}{24}.$$

The moment at the support is twice this, therefore it is the one that should be taken as the design moment.

The construction of a shearing force diagram demands no explanation: the diagram will be exactly the same as for a simply supported beam. The equation of the elastic curve of the beam is

$$EIy = \frac{ql}{2}\frac{x^3}{6} - \frac{ql^2}{12}\frac{x^2}{2} - \frac{qx^4}{24}.$$

The maximum deflection of the beam at $x = l/2$ is

$$y_{\max} = f_C = \frac{l}{EI}\left(\frac{ql^4}{96} - \frac{ql^4}{96} - \frac{ql^4}{384}\right) = -\frac{ql^4}{384EI}.$$

On p. 241 we found the maximum deflection at mid-span for a simply supported beam subjected to a uniformly distributed load.

It is given by

$$y_{\max} = -\frac{5ql^4}{384EI}.$$

Thus, from comparison of the deflections it is seen that in the case of a uniformly distributed load the deflection at mid-span of a beam with fixed ends is one-fifth that of a simply supported beam.

Example 72. A beam AB is fixed at both ends and carries a concentrated force P (Fig. 169a). Draw a bending moment and a shearing force diagram.

Solution. Set up equations for determining reactions. From the equilibrium conditions we have

$$\sum Y = 0, \qquad A + B = P;$$
$$\sum M_A = 0, \quad -Bl + m_B + Pa - m_A = 0.$$

From the condition that the slope of the tangent and the deflection over the support at B are zero, we obtain

$$-m_A l + A\frac{l^2}{2} - P\frac{b^2}{2} = 0,$$
$$-m_A\frac{l^2}{2} + A\frac{l^3}{6} - P\frac{b^3}{6} = 0.$$

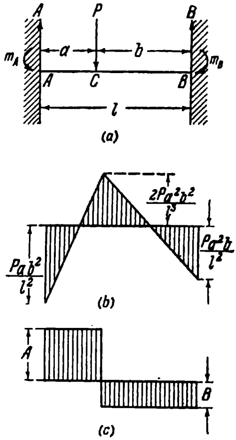


Fig. 169

Solving this system of equations, we find

$$A = P\frac{b^2(l+2a)}{l^3}, \qquad B = P\frac{a^2(l+2b)}{l^3},$$
$$m_A = P\frac{ab^2}{l^2}, \qquad m_B = P\frac{a^2b}{l^2}.$$

It is easy now to plot a bending moment and a shearing force diagram. The bending moment at section C where the force P is applied is

$$M_C = -m_A + Aa = -P\frac{ab^2}{l^2} + P\frac{ab^2(l+2a)}{l^3} = \frac{2Pa^2b^2}{l^3}.$$

The moment and shearing force diagrams are shown in Fig. 169b and c.

Determine the deflection at section C where the force P is applied. The elastic curve equation for the first portion of the beam is

$$EIy = -m\frac{x^2}{2} + A\frac{x^3}{6}.$$

Substituting the values of m, A and a for x, we obtain

$$y_C = \frac{1}{EI}\left[-\frac{Pab^2}{l^2}\frac{a^2}{2} + \frac{Pb^2(l+2a)}{l^3}\frac{a^3}{6}\right] = -\frac{Pa^3b^3}{3l^3EI}.$$

In the particular case when the force P acts at mid-length, i.e., for $a=b=l/2$, we have:

reactions

$$A = B = \frac{P}{2};$$

moments at supports

$$m_A = m_B = \frac{Pl}{8}. \quad (11.4)$$

These moments are negative as may be inferred from their senses indicated in Fig. 169 *a*.

The moment at mid-length is

$$M_{x=l/2} = \frac{Pl}{8}.$$

This moment is equal in magnitude to the moments at the supports but has an opposite sign.

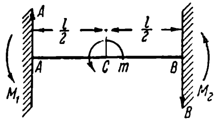


Fig. 170

Thus, the beam under consideration has three sections at which the bending moments are equal in absolute value, i. e., the beam has three equally dangerous sections.

The deflection at mid-length is

$$y_{x=l/2} = -\frac{Pl^3}{192EI}.$$

The maximum bending moment of a simply supported beam subjected to a force P at mid-length is $M_{x=l/2} = \frac{Pl}{4}$ (see p. 189) and the maximum deflection is $y_{x=l/2} = -\frac{Pl^3}{48EI}$ (see p. 239).

Consequently, compared to this beam, a beam fixed at both ends has half the bending moment and one-fourth the deflection at mid-span.

Example 73. A beam AB is fixed at both ends and subjected to a moment m at mid-span (Fig. 170). Determine the moments at the fixed ends.

Solution. Based on the conditions that the slope of the tangent and the deflection over the support at B are zero, we obtain from Eqs. (10.4) and (10.5)

$$-M_1 l + A\frac{l^2}{2} - \frac{ml}{2} = 0,$$

$$-M_1\frac{l^2}{2} + A\frac{l^3}{6} - \frac{ml^2}{8} = 0.$$

Solving this system of equations, we find

$$A = \frac{3}{2}\frac{m}{l}, \qquad M_1 = \frac{m}{4}.$$

From the condition $\Sigma M_B = 0$ we have

$$-\frac{m}{4} + \frac{3}{2}\frac{m}{l}l - m + M_2 = 0,$$

whence

$$M_2 = -\frac{m}{4}.$$

80. A Beam on Three Supports

Beams with more than two supports are called *continuous* or *multispan beams*. We shall consider the simplest continuous beam, viz. a beam on three supports.

Assume one support to be immovable and the other two on rollers, then for a beam subjected to forces perpendicular to the axis the supports will involve only vertical reactions.

A beam on three supports will have three unknown reactions two of which are determined from the conditions of static equilibrium and one reaction represents a redundant unknown. The redundant unknown is taken as the reaction arising at the middle support of the beam. The additional condition which is imposed on the deformation of the beam by the redundant unknown is that the deflection at the section over the middle support is zero. From this condition we shall determine the reaction at the middle support. After this reaction is found, the problem becomes statically determinate and its further solution presents no difficulty.

By way of illustration we consider a beam on three supports, which is subjected to a uniformly distributed load of intensity q (Fig. 171a). Determine the reactions at the supports.

Remove the middle support C. We then obtain a simply supported beam (Fig. 171b). Determine the deflection at the section over the removed support. This deflection is (see p. 241)

$$y_C = -\frac{5q(2l)^4}{384EI}.$$

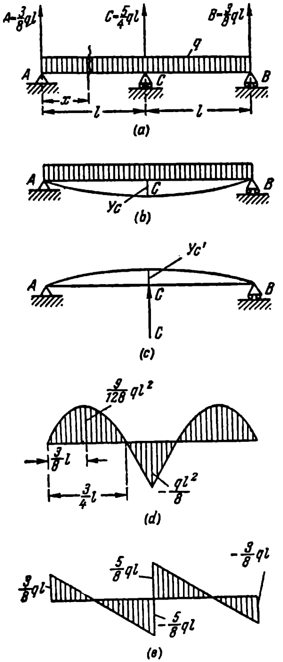


Fig. 171

The deflection of the beam (Fig. 171*c*) due to an upward concentrated force C is (see p. 239)

$$y_C' = \frac{C(2l)^3}{48EI}.$$

From the equation

$$y_C + y_C' = -\frac{5q(2l)^4}{384EI} + \frac{C(2l)^3}{48EI} = 0$$

we find the reaction C

$$C = \frac{5}{4}ql. \qquad (11.5)$$

When both spans have the same length, the reactions at the extreme supports A and B are, because of symmetry,

$$A = B = \frac{1}{2}\left(2ql - \frac{5}{4}ql\right) = \frac{3}{8}ql.$$

The M and Q diagrams are shown in Fig. 171*d* and *e*.

Example 74. A beam on three supports is subjected to a concentrated force P (Fig. 172*a*). Determine the reaction arising at the middle support.

Solution. Remove the middle support C; we then obtain a simply supported beam (Fig. 172*b*).

Determine the deflection at the section over the removed support. This deflection is equal to

$$EIy_C = -\frac{Pa\,[(l_1+l_2)^2 - a^2]\,l_1}{6(l_1+l_2)} + \frac{Pal_1^3}{6(l_1+l_2)},$$

$$y_C = -\frac{Pal_1\,[(l_1+l_2)^2 - a^2 - l_1^2]}{6(l_1+l_2)\,EI}.$$

Take now a beam (Fig. 172*c*) simply supported at its ends, A and B, and subjected to an upward force C. Determine the deflection at the point of application of the force C (see p. 239)

$$y_C = \frac{Cl_1^2 l_2^2}{3(l_1+l_2)\,EI}.$$

From the condition

$$y_C + y_C' = 0$$

we determine the reaction C

$$-\frac{Pal_1\left[(l_1+l_2)^2-a^2-l_1^2\right]}{6(l_1+l_2)EI}+\frac{Cl_1^2l_2^2}{3(l_1+l_2)EI}=0,$$

whence

$$C=\frac{Pa\left[(l_1+l_2)^2-a^2-l_1^2\right]}{2l_1l_2^2}. \quad (11.6)$$

If the force P acts in the left-hand span of the beam, the reaction at the middle support is determined from the same formula (11.6) but the distance a is then measured from the left support A and the spans l_1 and l_2 are interchanged.

In the particular case when the spans are of the same length, i. e., $l_1=l_2=l$, formula (11.6) is simplified. In this case the reaction C is

$$C=\frac{Pa(3l^2-a^2)}{2l^3}. \quad (11.7)$$

After determining the middle reaction from consideration of the deformation of the beam the other two reactions can easily be found from the equilibrium equation.

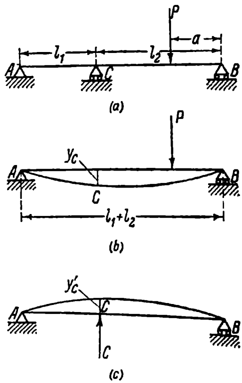


Fig. 172

Example 75. Choose an I-section for a two-span continuous beam (Fig. 173*a*) with equal spans $l=2$ m; the beam is loaded by a concentrated force $P=2$ tons at the middle section of the left-hand span; the right-hand span is subjected to a continuous load of intensity $q=4$ tons/m. The allowable stress $[\sigma]=1,400$ kgf/cm².

Solution. Remove the middle support C and determine the deflection under the removed support (Fig. 173*b*). Find first the reactions of the simply supported beam AB

$$\Sigma M_A=0, \qquad -B'2l+ql\left(l+\frac{l}{2}\right)+P\frac{l}{2}=0,$$

$$B'=\frac{3}{4}ql+\frac{1}{4}P;$$

$$\Sigma M_B=0, \qquad A'2l-P\left(l+\frac{l}{2}\right)-q\frac{l^2}{2}=0,$$

$$A'=\frac{ql}{4}+\frac{3}{4}P.$$

At the origin (point A) $f_0=0$ and $\alpha_0\neq 0$. Find $EI\alpha_0$ from the condition of zero deflection over support B. For $x=2l$ we obtain

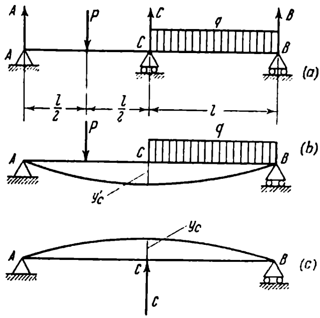


Fig. 173

from Eq. (10.5)

$$0=EI\alpha_0 2l+A'\frac{(2l)^3}{6}-P\frac{\left(2l-\frac{l}{2}\right)^3}{6}-q\frac{(2l-l)^4}{24},$$

whence

$$EI\alpha_0=-\frac{7}{32}Pl^2-\frac{7}{48}ql^3.$$

The deflection over the removed support C is determined from Eq. (10.5) at $x=l$

$$EIy'_C=EI\alpha_0 l+A'\frac{l^3}{6}-P\frac{\left(l-\frac{l}{2}\right)^3}{6}.$$

Substituting the values of A' and $EI\alpha_0$, we obtain

$$EIy'_C=-\frac{7}{32}Pl^3-\frac{7}{48}ql^4+\frac{ql^4}{24}+\frac{1}{8}Pl^3-\frac{Pl^3}{48}=-\frac{11}{96}Pl^3-\frac{5}{48}ql^4.$$

Write now the deflection of the beam (Fig. 173c) at section C due to the concentrated force C

$$EIy_C=C\frac{(2l)^3}{48}.$$

From the equation

$$EIy_C+EIy'_C=0$$

or

$$-\frac{11}{96}Pl^3-\frac{5}{48}ql^4+C\frac{l^3}{6}=0$$

we obtain the value of the reaction at the middle support

$$C=\frac{11}{16}P+\frac{5}{8}ql=\frac{11}{16}2+\frac{5}{8}\times4\times2=6\frac{3}{8}\text{ tons.}$$

Find now the reactions A and B for the given beam

$$\sum M_A=0,\qquad -B2l+ql\left(l+\frac{l}{2}\right)-\left(\frac{11}{16}P+\frac{5}{8}ql\right)l+P\frac{l}{2}=0,$$

$$B=\frac{7}{16}ql-\frac{3}{32}P=\frac{7}{16}\times4\times2-\frac{3}{32}\times2=3\frac{5}{16}\text{ tons;}$$

$$\sum M_B=0,\qquad A2l-P\frac{3l}{2}+\left(\frac{11}{16}P+\frac{5}{8}ql\right)l-\frac{ql^2}{2}=0,$$

$$A=\frac{13}{32}P-\frac{1}{16}ql=\frac{13}{32}\times2-\frac{1}{16}\times4\times2=\frac{5}{16}\text{ ton.}$$

Draw a bending moment diagram.
In the first portion

$$M_1=Ax=\frac{5}{16}x.$$

At $x=0$,

$$M_A=0;$$

at $x=\frac{l}{2}=1$ m,

$$M=\frac{5}{16}\text{ ton-m.}$$

In the second portion

$$M_2=Ax-P\left(x-\frac{l}{2}\right)=\frac{5}{16}x-2(x-1).$$

At $x=\frac{l}{2}=1$ m,

$$M=\frac{5}{16}\text{ ton-m;}$$

at $x=l=2$ m,

$$M_C=\frac{5}{16}\times2-2\times1=-1\frac{3}{8}\text{ tons-m.}$$

The moments in the third portion are determined proceeding from the right end of the beam

$$M_3=Bx-q\frac{x^2}{2}=3\frac{5}{16}x-2x^2.$$

At $x=0$,

$$M_B=0;$$

at $x=l=2$ m,

$$M_C = 3\frac{5}{16}\times 2 - 2\times 2^2 = -1\frac{3}{8}\text{ tons-m.}$$

Find the maximum value of the moment in the third portion

$$\frac{dM_3}{dx} = 0 = 3\frac{5}{16} - 4x, \qquad \text{hence } x = \frac{53}{64}\text{ m.}$$

For this value of x, the moment is

$$M_{3\,max} = 3\frac{5}{16}\frac{53}{64} - 2\left(\frac{53}{64}\right)^2 = 1\frac{761}{2,048}\text{ tons-m.}$$

This bending moment is numerically slightly less than M_C. Therefore, the moment over support C is the maximum moment over the length of the beam; it is equal to $M_C = -1\frac{3}{8}$ tons-m. Substituting this value of the moment in the design equation, we obtain the required value of Z

$$Z \geqslant \frac{M_C}{[\sigma]} = \frac{1,100,000}{8\times 1,400} = 98.2\text{ cm}^3.$$

The nearest larger value of Z is found in a No. 16 I-beam, viz. 109 cm^3.

81. Check Questions

What beams are continuous? statically indeterminate?

What methods are used to solve statically indeterminate beams?

What is the ratio between the maximum deflections of two beams subjected to a uniformly distributed load if one is simply supported at both ends and the other is fixed at the ends?

What are the moments at the two fixed ends of a beam subjected to a couple at mid-length? to a force P at mid-length? to a load uniformly distributed along the length of the beam?

State the principal advantages of statically indeterminate beams over statically determinate ones.

Chapter XII

Complex resistance

82. Oblique Bending

So far the consideration has been restricted to plane bending when the plane of action of loads coincides with a longitudinal plane of symmetry of a beam or, in general, with one of its principal planes. The bending deformation then takes place in the plane of action of moments and the neutral axis coincides with a principal axis of inertia of the cross section and is perpendicular to the plane of action of moments.

There are certain cases, however, where the plane of action of bending moments does not coincide with any one of the principal planes of a beam. Such bending is called *oblique bending*.

Consider an example of oblique bending. Suppose that a beam of rectangular section is fixed at one end (Fig. 174a and b) and carries a force P acting perpendicular to the beam axis at its free end and making an angle α with the principal xy plane. Since the plane of action of the bending moment does not coincide with either of the principal planes of the beam, we have oblique bending in this case.

The absolute value of the bending moment at a section mn a distance x from the fixed end is

$$M = P(l - x).$$

We resolve the force P into two components, P_y and P_z, acting along the principal y and z axes of the section. The absolute values of the component moments are then

$$M_z = P_y(l - x) = P(l - x)\cos\alpha,$$
$$M_y = P_x(l - x) = P(l - x)\sin\alpha.$$

The moments M_y and M_z act in the principal planes of the beam. We know how to determine the stresses and deflections due to either of these moments acting separately. Using the principle of superposition, we can find the stresses and deflections resulting from a simultaneous action of the moments M_y and M_z. Thus, *the case of oblique bending can always be reduced to two plane or, as is sometimes said, simple bendings.*

When only one moment, M_z, is acting, the neutral axis is the z axis (Fig. 174c) and the normal stress for a point N of co-ordinates z, y in the first quadrant of the section mn is defined by for-

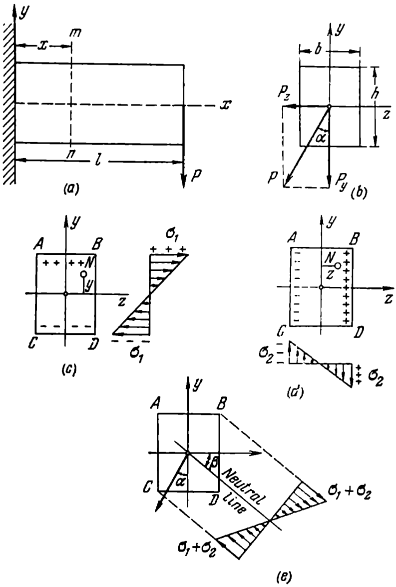


Fig. 174

mula (9.3)

$$\sigma_1 = \frac{M_z y}{I_z}.$$

The stress at the same point due to the action of the moment M_y alone (Fig. 174d) is

$$\sigma_2 = \frac{M_y z}{I_y}.$$

When the two moments, M_y and M_z, are acting simultaneously, the stress at any point in the section is equal to the algebraic sum

of the stresses σ_1 and σ_2, i. e.,

$$\sigma = \sigma_1 + \sigma_2 = \frac{M_z y}{I_z} + \frac{M_y z}{I_y}. \tag{12.1}$$

The co-ordinates y, z of points of the section and the bending moments M_y and M_z are substituted in this formula with the proper signs. If the moment acts so that it produces tension in the first quadrant where the co-ordinates z and y are positive, it is given a plus sign, otherwise a minus sign. In our case, for instance, both moments, M_y and M_z, are positive since they produce tension in the first quadrant; we thus obtain:

for point $A\left(z = -\frac{b}{2},\ y = \frac{h}{2}\right)$

$$\sigma = \frac{P(l-x)\cos\alpha \frac{h}{2}}{I_z} - \frac{P(l-x)\sin\alpha \frac{b}{2}}{I_y},$$

for point $B\left(z = \frac{b}{2},\ y = \frac{h}{2}\right)$

$$\sigma = \frac{P(l-x)\cos\alpha \frac{h}{2}}{I_z} + \frac{P(l-x)\sin\alpha \frac{b}{2}}{I_y},$$

for point $C\left(z = -\frac{b}{2},\ y = -\frac{h}{2}\right)$

$$\sigma = \frac{-P(l-x)\cos\alpha \frac{h}{2}}{I_z} - \frac{P(l-x)\sin\alpha \frac{b}{2}}{I_y},$$

for point $D\left(z = \frac{b}{2},\ y = -\frac{h}{2}\right)$

$$\sigma = \frac{-P(l-x)\cos\alpha \frac{h}{2}}{I_z} + \frac{P(l-x)\sin\alpha \frac{b}{2}}{I_y}.$$

The maximum overall stress occurs at points B and C in this case: the tensile stress at point B ($x > 0$, $y > 0$) and the compressive stress at point C ($x < 0$, $y < 0$). The absolute values of these stresses are the same.

The equation of the neutral line is obtained by setting the right-hand side of formula (12.1) equal to zero

$$\frac{M_z y}{I_z} + \frac{M_y z}{I_y} = 0$$

or

$$\frac{My\cos\alpha}{I_z} + \frac{Mz\sin\alpha}{I_y} = 0,$$

whence

$$\frac{y\cos\alpha}{I_z}+\frac{z\sin\alpha}{I_y}=0.$$

This equation of a straight line is satisfied by the values $y=0$ and $z=0$; consequently, *the neutral line passes through the centroid of the cross section.*

After determining the ratio y/z from the last expression we find the tangent of the angle β that the neutral line makes with the positive z axis (Fig. 174*e*)

$$\tan\beta=\frac{y}{z}=-\tan\alpha\frac{I_z}{I_y}. \tag{12.2}$$

From formula (12.2) it is seen that for sections in which $I_z=I_y$ (square, circle, etc.) *the neutral line is always perpendicular to the plane of action of the bending moment*, the plane in which the bending deformation takes place, i. e., no oblique bending can occur in beams in which all centroidal axes of cross sections (regular figures) are principal.

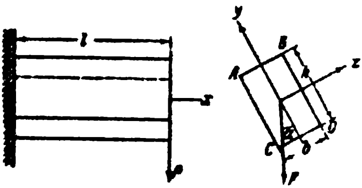


Fig. 175

In cases where $I_z\neq I_y$ and $\alpha\neq 0$ or $\alpha\neq 90°$ the neutral line is not perpendicular to the plane of the bending moment.

Example 76. A beam of rectangular section bh is fixed at one end (Fig. 175) and bent by a force $P=1{,}200$ kgf applied at the free end and making an angle $\alpha=30°$ with the principal xy plane. Determine the dimensions of the section if the length of the beam is $l=1$ m, $b=0.6\,h$ and the allowable stress $[\sigma]=1{,}200$ kgf/cm².

Solution. The maximum bending moment at the fixed end is

$$M_{max}=Pl=1{,}200\times 100=120{,}000 \text{ kgf-cm}.$$

We resolve this moment, acting in the vertical plane, along the principal y and z axes of the section

$$M_z=120{,}000\cos 30°=120{,}000\times 0.866=104{,}000 \text{ kgf-cm},$$
$$M_y=120{,}000\sin 30°=120{,}000\times 0.5=60{,}000 \text{ kgf-cm}.$$

The maximum stresses defined by formula (11.4) occur at points B and C. They are equal in absolute value. Determine the stress at point B $\left(y=\frac{h}{2},\ z=\frac{b}{2}\right)$

$$\sigma=\frac{M_z\frac{h}{2}}{I_z}+\frac{M_y\frac{b}{2}}{I_y}=\frac{6M_z}{bh^2}+\frac{6M_y}{hb^2}.$$

Substituting the values of the moments M_y and M_z and equating the absolute value of the maximum stress to the allowable stress, we obtain

$$1{,}200=\frac{6\times 104{,}000}{bh^2}+\frac{6\times 60{,}000}{hb^2}$$

or

$$1=\frac{520}{bh^2}+\frac{300}{hb^2}.$$

Substituting $b=0.6h$, we obtain an equation for determining the depth of the section h

$$1=\frac{520}{0.6h^3}+\frac{300}{0.36h^3},$$

whence

$$h=\sqrt[3]{\frac{520}{0.6}+\frac{300}{0.36}}=\sqrt[3]{1{,}700}=11.9\text{ cm}.$$

We round off this value to $h=12$ cm; then

$$b=0.6\times 12=7.2\text{ cm}.$$

For these dimensions of the section, the maximum stress in the beam is

$$\sigma=\frac{104{,}000}{\frac{7.2\times 12^2}{6}}+\frac{60{,}000}{\frac{12\times 7.2^2}{6}}=600+580=1{,}180\text{ kgf/cm}^2<[\sigma].$$

Example 77. A No. 14 Z-beam is simply supported on two trusses and carries a uniformly distributed vertical load of intensity $q=400$ kgf/m (Fig. 176*a* and *b*). Determine the normal stresses at points A, B, C and D in the dangerous section and the maximum deflection if the distance between the trusses is $l=2$ m, the inclination of the truss to the horizon is 30° and $E=2\times 10^6$ kgf/cm^2.

Solution. From tables on standard rolled sections, for a No. 14 Z-section we take the principal moments of inertia $I_{z_1}=759$ cm^4, $I_{y_1}=130.7$ cm^4 and the angle of inclination of the principal axes $\alpha=14°27'$ (Fig. 176*c*).

The maximum bending moment in the beam is at mid-length (8.8)

$$M_{max}=\frac{ql^2}{8}=\frac{400\times 2^2}{8}=200\text{ kgf-m}=20{,}000\text{ kgf-cm}.$$

Resolving the moment M_{max}, acting in the vertical plane, along the principal axes of the section, we find

$$M_{z_1} = -M_{max} \cos (30° - 14°27') = \\ = -20{,}000 \times 0.9634 = -19{,}268 \text{ kgf-cm},$$
$$M_{y_1} = -M_{max} \sin (30° - 14°27') = \\ = -20{,}000 \times 0.2681 = -5{,}362 \text{ kgf-cm}.$$

The moments M_{y_1} and M_{z_1} are negative since they produce compression in the first quadrant.

To determine the stresses at points A, B, C and D we calculate their distances to the principal z_1 and y_1 axes.

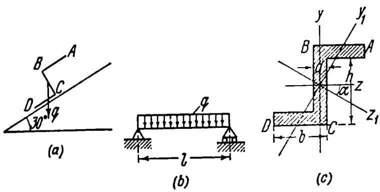


Fig. 176

The co-ordinates of point A with respect to the principal z_1 and y_1 axes are determined by applying the formulas of transformation of co-ordinates when the axes are rotated

$$z_1 = y \sin (-14°27') + z \cos (-14°27') = \\ = 7(-0.2495) + 6.1 \times 0.9684 = 4.15 \text{ cm},$$
$$y_1 = y \cos (-14°27') - z \sin (-14°27') = \\ = 7 \times 0.9684 - 6.1(-0.2495) = 8.29 \text{ cm}.$$

The co-ordinates of point D with respect to the z_1 and y_1 axes are

$$z_1 = -4.15 \text{ cm}, \qquad y_1 = -8.29 \text{ cm}.$$

The co-ordinates of point B with respect to the z and y axes are

$$z = -\frac{d}{2} = -\frac{0.8}{2} = -0.4 \text{ cm}, \qquad y = \frac{h}{2} = 7 \text{ cm}.$$

The co-ordinates of point B with respect to the principal z_1 and y_1 axes are determined by the formulas of transformation of co-or-

dinates when the axes are rotated

$$z_1 = 7\sin(-14°27') + (-0.4)\cos(-14°27') =$$
$$= -7\times 0.2495 - 0.4\times 0.9684 = -2.14 \text{ cm},$$
$$y_1 = 7\cos(-14°27') - (0.4)\sin(-14°27') =$$
$$= 7\times 0.9684 - 0.4\times 0.2495 = 6.67 \text{ cm}.$$

The co-ordinates of point C with respect to the z_1 and y_1 axes are

$$z_1 = 2.14 \text{ cm}, \qquad y_1 = -6.67 \text{ cm}.$$

We now determine the stresses at points A, B, C and D. The stress at point A is

$$\sigma_A = \frac{-19{,}268 \times 8.29}{759} + \frac{-5{,}366\times 4.15}{130.7} = 380 \text{ kgf/cm}^2.$$

At point D

$$\sigma_D = \frac{-19{,}268(-8.29)}{759} + \frac{-5{,}366(-4.15)}{130.7} = -380 \text{ kgf/cm}^2.$$

At point B

$$\sigma_B = \frac{-19{,}268 \times 6.67}{759} + \frac{-5{,}366(-2.14)}{130.7} = -81 \text{ kgf/cm}^2.$$

At point C

$$\sigma_C = \frac{-19{,}268(-6.67)}{759} + \frac{-5{,}366 \times 2\times 14}{130.7} = 81 \text{ kgf/cm}^2.$$

The maximum tensile stress at point A and the maximum compressive stress at point D are each equal to 380 kgf/cm².

The maximum deflection of the beam occurs at mid-length. To calculate it we resolve the uniformly distributed load, acting in the vertical plane, along the principal axes of the section

$$q_y = q\cos(30° - 14°27') = 400\times 0.9634 = 385 \text{ kgf/m} = 3.85 \text{ kgf/cm},$$
$$q_z = q\sin(30° - 14°27') = 400\times 0.2681 = 107 \text{ kgf/m} = 1.07 \text{ kgf/cm}.$$

The deflections in the principal planes are, according to formula (10.17),

in the yx plane

$$f_y = -\frac{5q_y l^4}{384EI_z},$$

in the zx plane

$$f_z = -\frac{5q_z l^4}{384EI_y}.$$

The total deflection is equal to the geometric sum of the deflections f_y and f_z, i. e.,

$$f = \sqrt{f_y^2 + f_z^2} = \frac{5\times 200^4}{384\times 2\times 10^6}\sqrt{\left(\frac{3.85}{759}\right)^2 + \left(\frac{1.07}{130.7}\right)^2} =$$
$$= 10.4\times 0.00965 = 0.1 \text{ cm}$$
$$f = 1 \text{ mm}.$$

83. Bending Combined with Tension or Compression

In the bending analysis of beams we have assumed so far that the external forces acting on a beam are perpendicular to its axis. Consider now a more general case when the bending load is inclined to the axis of a beam. Suppose, for example, that a beam fixed at one end (Fig. 177*a*) is acted on by a force P in a longi-

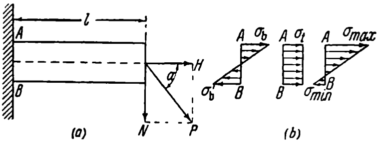


Fig. 177

tudinal plane of symmetry of the beam at an angle α to the beam axis.

We resolve the force P into two components, N and H. The force N acting perpendicular to the axis of the beam produces bending, and the force H acting along the axis produces tension.

The normal stress caused by the tensile force H is the same at all cross sections of the beam and is uniformly distributed over the section. The magnitude of this stress is given by the formula

$$\sigma_t = \frac{H}{A},$$

where A is the cross-sectional area of the beam.

The bending stresses depend on the magnitude of the moment. The maximum bending moment occurs at the fixed end; therefore, the most dangerous section is the one adjacent to the fixed end. The maximum stresses at this section in the fibres most remote from the neutral layer are

$$\sigma_b = \frac{Nlh_1}{I}, \qquad \sigma_b' = -\frac{Nlh_2}{I}, \tag{12.3}$$

where σ_b is the tensile stress in the top extreme fibres, σ_b' is the compressive stress in the bottom extreme fibres, h_1 and h_2 are the distances of the extreme fibres from the neutral line, I is the moment of inertia of the whole section with respect to the neutral line.

The overall stress due to bending and tension for point A is

$$\sigma_{max} = \frac{H}{A} + \frac{Nlh_1}{I}, \tag{12.4}$$

for point B

$$\sigma_{min}=\frac{H}{A}-\frac{Nlh_2}{I}. \tag{12.5}$$

The stress σ_{min} may be a tensile stress if $\frac{H}{A}>\frac{Nlh_2}{I}$, a compressive stress if $\frac{H}{A}<\frac{Nlh_2}{I}$ and, finally, σ_{min} may be zero if $\frac{H}{A}=\frac{Nlh_2}{I}$. Thus, the sign of the stress σ_{min} depends on the relation between the stresses $\frac{H}{A}$ and $\frac{Nlh_2}{I}$.

In the particular case when $h_1=h_2$ we have

$$\sigma_{max}=\frac{H}{A}+\frac{Nl}{Z}, \tag{12.6}$$

$$\sigma_{min}=\frac{H}{A}-\frac{Nl}{Z}. \tag{12.7}$$

Figure 177*b* shows stress diagrams for the case when $|\sigma_b'|>\sigma_t$. If the force H tends to compress and not stretch the beam, then similar reasoning will lead to the following formulas for determining the total stresses

$$\sigma_{max}=-\frac{H}{A}+\frac{Nlh_1}{I}, \tag{12.8}$$

$$\sigma_{min}=-\frac{H}{A}-\frac{Nlh_2}{I}. \tag{12.9}$$

In this case it is assumed that the beam is so stiff and deflects so slightly that the compressive force H is always acting parallel to the axis of the beam producing no bending. This is also taken account of in formulas (12.4) and (12.5).

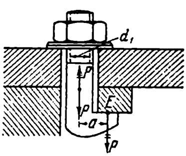


Fig. 178

Example 78. Determine the stress in the bolt shown in Fig. 178 if the inner diameter of the bolt thread is $d_1=25.138$ mm, the tightening force $P=400$ kgf and the eccentricity of the load $a=50$ mm.

Solution. Since the bolt head is non-symmetrical, the reaction P due to the tightening force acts on the bolt head not at its centre but at the centre of the bearing surface of the head, at point E. The force P stretches the bolt and produces a bending moment Pa.

The bending stress in the bolt is

$$\sigma_b=\frac{M}{Z}=\frac{Pa}{\frac{\pi d_1^3}{32}}=\frac{32Pa}{\pi d_1^3}.$$

The tensile stress in the bolt is

$$\sigma_t = \frac{P}{A} = \frac{P}{\frac{\pi d_1^2}{4}} = \frac{4P}{\pi d_1^2}.$$

The total stress due to bending and tension at the most dangerous point is

$$\sigma_{max} = \frac{32Pa}{\pi d_1^3} + \frac{4P}{\pi d_1^2} = \frac{4P}{\pi d_1^2}\left(\frac{8a}{d_1} + 1\right).$$

Substituting the values of P, a and d_1, we obtain

$$\sigma_{max} = \frac{4\times 400}{3.14\times 2.5138^2}\left(\frac{8\times 5}{2.5138} + 1\right) \cong 1{,}360 \text{ kgf/cm}^2.$$

84. Eccentric Compression

By eccentric compression is meant the case when the force compressing a rod is parallel to the rod axis and lies in one of the principal planes but its point of application does not coincide with the centroid of the section. It is assumed that the dimensions of the rod are such that the deflection of the rod axis from its original position is so small in comparison with the eccentricity that it may be neglected.

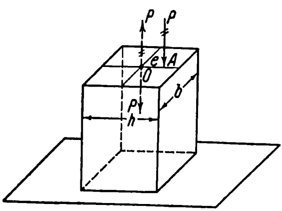


Fig. 179

This case of compression is shown in Fig. 179. The compressive longitudinal force acts on the rod at point A with eccentricity e. We apply two equal and opposite forces P at the centroid O of the top cross section. Then the force P acting at point A and the upward force P acting at point O give a couple of moment Pe. This moment acts in the principal plane of the rod and remains constant throughout its length. The stress produced by it is

$$\sigma_b = \pm \frac{Pe}{Z}.$$

The remaining force P acting along the axis of the rod produces a compressive stress equal to

$$\sigma_c = -\frac{P}{A}.$$

The total stress is equal to the algebraic sum of the stresses σ_b and σ_c

$$\sigma = \sigma_c + \sigma_b$$

or, substituting the values of σ_c and σ_b, we obtain

$$\sigma_{max} = -\frac{P}{A} + \frac{Pe}{Z}, \qquad (12.10)$$

$$\sigma_{min} = -\frac{P}{A} - \frac{Pe}{Z}. \qquad (12.11)$$

Numerically σ_{min} is larger than σ_{max}. The formulas for determining σ_{max} and σ_{min} in eccentric compression are the same as for combined bending and compression treated in the preceding section since eccentric compression reduces to a combination of bending and compression.

85. The General Case of Eccentric Compression or Tension

Consider now the case when a compressive or tensile longitudinal force does not lie in either of the principal planes of a rod. Suppose that the rod shown in Fig. 180 is compressed by a force P

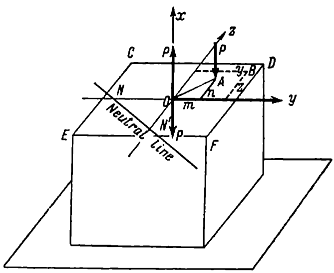


Fig. 180

parallel to the rod axis. The point of application A of this force does not lie in the principal planes of the rod, zOx or yOx.

At the centroid O of the top base we apply two equal forces P directed along the x axis in an opposite sense. Then the force P acting at point A and the upward force P acting at point O give a couple of moment $P\,\overline{AO}$ tending to bend the rod in the plane AOx.

The downward force P acting at point O produces compressive stresses $-\frac{P}{A}$ in the rod, where A is the cross-sectional area. Thus, the general case of eccentric compression (tension) reduces to combined oblique bending and simple compression (tension). Let the co-ordinates of point A be m and n. We find the stress at a point B of co-ordinates y and z. Resolve the moment $P\,\overline{AO}$ acting in the plane AOx into two moments acting in the principal planes zOx and yOx. We then obtain a moment Pn in the plane zOx and a moment Pm in the plane yOx. Both moments produce compressive stresses in the first quadrant containing point B

$$-\frac{Pnz}{I_y} \quad \text{and} \quad -\frac{Pmy}{I_z}.$$

The total stress at point B is found by adding the three stresses

$$\sigma = -\left(\frac{P}{A}+\frac{Pnz}{I_y}+\frac{Pmy}{I_z}\right). \tag{12.12}$$

For many cross sections with salient angles in which both principal axes are axes of symmetry (rectangle, I-section, etc.) it is easy to determine a point where the stress is maximum. In the case under consideration the most highly stressed point is D. The stress at point D is

$$\sigma = -\left(\frac{P}{A}+\frac{Pn}{Z_y}+\frac{Pm}{Z_z}\right). \tag{12.13}$$

For other vertices of the rectangle the stresses are

for point C

$$\sigma = -\frac{P}{A}-\frac{Pn}{Z_y}+\frac{Pm}{Z_z},$$

for point E

$$\sigma = -\frac{P}{A}+\frac{Pn}{Z_y}+\frac{Pm}{Z_z},$$

for point F

$$\sigma = -\frac{P}{A}+\frac{Pn}{Z_y}-\frac{Pm}{Z_z}.$$

If the section of a rod is of arbitrary shape, the determination of the most highly stressed point of the section requires a knowledge of the position of the neutral line (also called zero line in this case).

The neutral line is the locus of points with zero stresses; its equation is obtained by setting the stress σ in formula (12.12) equal to zero. We have then

$$\frac{P}{A}+\frac{Pnz}{I_y}+\frac{Pmy}{I_z}=0$$

or, factoring out $\frac{P}{A}$ and dividing both sides by it $\left(\frac{P}{A} \neq 0\right)$, we obtain

$$1+\frac{Anz}{I_y}+\frac{Amy}{I_z}=0. \tag{12.14}$$

The moment of inertia may be represented as the product

$$I=Ai^2. \tag{12.15}$$

The quantity i is called the *radius of gyration of the section.* It will be recalled that the moment of inertia is expressed in the general form as

$$I=\int_A y^2\,dA,$$

where y is the distance of elementary areas of the section to the axis with respect to which the moment of inertia is calculated.

From formula (12.15) we have

$$i_y=\sqrt{\frac{I_y}{A}}, \qquad i_z=\sqrt{\frac{I_z}{A}}. \tag{12.16}$$

On the basis of formula (12.16) we can always calculate the radius of gyration of a section with respect to any axis if the moment of inertia of the section with respect to that axis and its area are known.

Substituting $\frac{1}{i_y^2}$ and $\frac{1}{i_z^2}$ for $\frac{A}{I_y}$ and $\frac{A}{I_z}$, respectively, in the equation of the neutral line (12.14), we obtain

$$1+\frac{nz}{i_y^2}+\frac{my}{i_z^2}=0.$$

Setting successively $z=0$ and $y=0$ in this equation, we find the intercepts of the neutral line on the y and z axes (see Fig. 180)

$$\left.\begin{aligned} ON&=y_0=-\frac{i_z^2}{m}, \\ ON'&=z_0=-\frac{i_y^2}{n}. \end{aligned}\right\} \tag{12.17}$$

After plotting the neutral line it is easy to find a point most remote from it. Taking the co-ordinates y and z of this point and substituting them in formula (12.12), we determine the maximum stress at the section.

86. Concept of Core of Section

In eccentric compression or tension the stresses arising at a cross section may be of opposite or the same sign, depending on the point of application of the force. If the force is applied at a point such that the neutral line passes through the section, the stresses on one side of it are compressive, and those on the other side are tensile. The position of the neutral line depends on the point of application of the force.

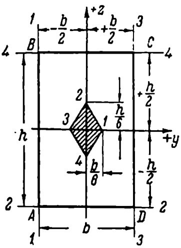


Fig. 181

In cases where an eccentrically compressed bar is made of a material which is weak in tension, such as cast iron, it is desirable to avoid tensile stresses at a section. This is also true for the design of masonry since brick and stone are weak in tension and, what is more, there will be a danger of opening of joints between separate bricks or stones if they are in tension.

To avoid tensile stresses at the section, the point of application of a compressive force for a given cross section must not be removed from the centroid of the section to a distance greater than a certain limiting value, i. e., the magnitude of eccentricity must be limited.

This limiting distance of the point of application of a force from the centroid of a section must correspond to a position of the neutral line such that it does not cross the contour of the section but just touches it. In this case the stresses acting at the section will be of the same sign. Thus, if we have a rectangular cross section *ABCD* (Fig. 181), the limiting positions of the neutral line are tangents to the contour of the section, i. e., lines 1-1, 2-2, 3-3, 4-4. For each of these four positions of the neutral line it is easy, as will be shown later, to find the corresponding points of application of the force. Let these points be 1, 2, 3, 4; joining them, we obtain the shaded quadrangle. If the point of application of a compressive force lies inside this quadrangle, there will be only compressive stresses throughout the cross section *ABCD*. The same conclusion may be drawn for a tensile force.

A confined area enclosing the centroid of a section (quadrangle 1-2-3-4 in our example) such that a force applied at any of its points produces stresses of the same sign throughout the section is called the *core* or *kern of the section*.

We now proceed to the method of plotting the contour of the core of the section. This method consists in finding the points of

application of a force that correspond to neutral lines touching the contour to the cross section.

As an example let us find the contour of the core for the rectangular cross section $ABCD$ shown in Fig. 181. The intercepts that the neutral lines in their limiting positions make on the co-ordinate axes are:

the intercept of neutral line 1-1 on y axis is $-\frac{b}{2}$

the intercept of neutral line 2-2 on z axis is $-\frac{h}{2}$

the intercept of neutral line 3-3 on y axis is $+\frac{b}{2}$

the intercept of neutral line 4-4 on z axis is $+\frac{h}{2}$

The point of application of the force on the y axis corresponding to the neutral line 1-1 is determined from formula (12.17)

$$y_0 = -\frac{i_z^2}{m};$$

since $y_0 = -\frac{b}{2}$ and

$$i_z^2 = \frac{I_z}{A} = \frac{hb^3}{12hb} = \frac{b^2}{12},$$

we have

$$m = \frac{b^2}{12} : \frac{b}{2} = \frac{b}{6}.$$

This point of application of the force, which is one of the vertices of the contour of the core, is marked 1 in Fig. 181. Because of symmetry it may be concluded that to the neutral line 3-3 corresponds the vertex of the contour of the core marked 3 in the figure. Similarly, it is easy to find the other two vertices, 2 and 4, of the contour of the core. Joining the vertices by straight lines, we obtain a rhombus 1-2-3-4 which represents the contour of the core for the rectangle.

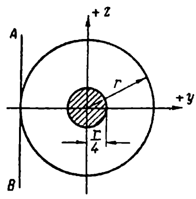


Fig. 182

Example 79. Find the core for the circle of radius r shown in Fig. 182.

Solution. Because of symmetry the contour of the core of the circle is a circumference. To determine the radius of this circumference we take some position of the neutral axis, such as a tangent AB perpendicular to the y axis (Fig. 182). The intercept y_0 of this neutral line on the y axis is equal to the radius, i.e., $y_0 = -r_0$.

Substitute this value of y_0 in formula (12.17)

$$-r=-\frac{i^2}{m},$$

whence the distance m of the point of application of the force from the centre of the circle, i. e., the radius of the core of the section, is

$$m=\frac{i^2}{r}=\frac{I}{Ar}.$$

Since $I=\frac{\pi r^4}{4}$ and $A=\pi r^2$, we have

$$m=\frac{\pi r^4}{4r\pi r^2}=\frac{r}{4}.$$

87. Combined Bending and Torsion

In practice torsion is rather frequently accompanied by bending. We have to deal with this complex type of deformation, for example, in the design of shafts when the forces transmitted to a shaft do not pass through its axis. Suppose, for instance, that a gear is mounted on a shaft (Fig. 183) and transmits a circumferential force P from a driving gear. Transfer the force P to the centre of the shaft O. To do this we apply two equal and opposite forces P at point O along a straight line parallel to the force P. We then obtain a couple of moment PR (the forces constituting this couple are marked by two dashes in the drawing) which tends to twist the shaft, and a force P applied at the centre of the shaft and tending to bend the shaft.

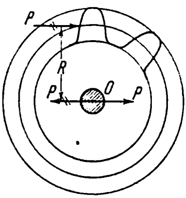


Fig. 183

On the basis of the previously derived formulas (9.7) and (6.10)

$$\sigma=\frac{M}{Z}, \tag{9.7}$$

$$\tau=\frac{M_t}{Z_p} \tag{6.10}$$

it is easy to determine the normal stress σ due to bending and the shearing stress τ due to torsion for any section of the shaft. The direct shearing stresses due to the shearing force are usually neglected in the design of shafts since these stresses are considerably smaller than the torsional shearing stresses due to the twisting moment.

The maximum torsional stresses and the maximum bending stresses occur on the surface of the shaft (Fig. 184a). Each of these

stresses taken separately may be smaller than the allowable stress for the corresponding type of deformation. However, their simultaneous effect may prove dangerous for the shaft.

In order to evaluate the simultaneous effect to the bending stress σ and the torsional stress τ we isolate an element of material (Fig. 184*b*) at the most dangerous point (point *a* or *b* in Fig. 184*a*) in the most dangerous section.

The four faces of this element are acted on by the shearing stresses and two of these faces are also acted on by the normal

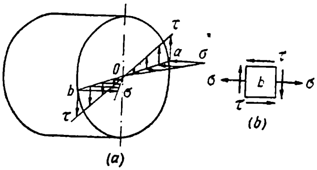


Fig. 184

stresses. This element is in a state of plane stress (see Sec. 68). The magnitudes of the three principal stresses for this element are

$$\sigma_1 = \frac{\sigma}{2} + \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2},$$

$$\sigma_2 = 0,$$

$$\sigma_3 = \frac{\sigma}{2} - \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2}.$$

Knowing the values of the principal stresses, we can write the strength condition using one or another strength theory. Thus, on the basis of the maximum shearing stress theory (the third strength theory)

$$\sigma_1 - \sigma_3 \leqslant [\sigma].$$

Substituting the values of σ_1 and σ_3, we obtain the following strength condition

$$\sqrt{\sigma^2 + 4\tau^2} \leqslant [\sigma]. \qquad (12.18)$$

The stress $\sqrt{\sigma^2 + 4\tau^2}$ is called the effective or equivalent stress. Substituting in (12.18) the values of σ and τ defined by formulas (10.5) and (6.10), we obtain

$$\sqrt{\left(\frac{M}{Z}\right)^2 + 4\left(\frac{M_t}{Z_p}\right)^2} \leqslant [\sigma]. \qquad (12.19)$$

Since $Z_p = 2Z$ for a circular and an annular section, we have

$$\frac{1}{Z}\sqrt{M^2 + M_t^2} \leqslant [\sigma]. \qquad (12.20)$$

Thus, the design formula for combined bending and torsion is similar to that for bending (10.5), the only difference being that the bending moment is replaced by some other moment, called the effective moment, of magnitude $\sqrt{M^2 + M_t^2}$.

The most dangerous section of a shaft is obviously that for which the magnitude of the effective moment is maximum. Therefore, in cases where it is difficult to locate the most dangerous section at once, an effective moment diagram is plotted.

If the bending forces do not lie in the same plane, each of the forces is first resolved along two directions, vertical and horizontal. Next, it is necessary to plot the bending moment (M_v) diagram due to the forces acting in the vertical plane and the M_h diagram due to the forces acting in the horizontal plane. The moments M_v and M_h are used to plot the total bending moment diagram. The total moments are determined by the formula

$$M = \sqrt{M_v^2 + M_h^2}. \qquad (12.21)$$

Substituting this expression for the bending moment in the design formula (12.20), we obtain

$$\frac{1}{Z}\sqrt{M_v^2 + M_h^2 + M_t^2} \leqslant [\sigma]. \qquad (12.22)$$

In this case the effective moment is $\sqrt{M_v^2 + M_h^2 + M_t^2}$.

The design formula based on the energy strength theory for the case of combined bending and torsion is obtained by substituting the values of σ_1, σ_2 and σ_3 in the strength condition (4.38). After substitution we have

$$\sqrt{\sigma^2 + 3\tau^2} \leqslant [\sigma]. \qquad (12.23)$$

Substituting the values of σ and τ in this formula, we obtain

$$\sqrt{\left(\frac{M}{Z}\right)^2 + 3\left(\frac{M_t}{Z_p}\right)^2} \leqslant [\sigma].$$

or, taking into account that $Z_p = 2Z$, we have

$$\frac{1}{Z}\sqrt{M^2 + 0.75M_t^2} \leqslant [\sigma]. \qquad (12.24)$$

Example 80. A pulley of weight 0.5 ton and diameter 1.2 m is mounted at the middle of a shaft driven by a motor M (Fig. 185). The tension in the tight part of the belt which passes over the pulley is 600 kgf, and the tension in the loose part is 300 kgf.

Determine the diameter of the shaft if the allowable stress is $[\sigma]=500$ kgf/cm².

Solution. Transferring the forces of tension in the belts to the centre of the shaft, we find that the shaft is acted on by a horizontal force

$$P_h = 600+300=900 \text{ kgf}$$

and a torque

$$M_t = 600\times 60-300\times 60 = 1{,}800 \text{ kgf-cm}.$$

The horizontal force $P_h=900$ kgf and the vertical force representing the weight of the pulley $P_v=500$ kgf act at the same sec-

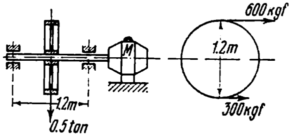


Fig. 185

tion of the shaft and their resultant is

$$P=\sqrt{P_h^2+P_v^2}=\sqrt{900^2+500^2}=1{,}030 \text{ kgf}.$$

The maximum bending moment due to this resultant occurs at the middle of the shaft and is equal to

$$M=\frac{Pl}{4}=\frac{1{,}030\times 120}{4}=30{,}900 \text{ kgf-cm}.$$

Substituting the values of M and M_t in the design formula (12.20), we obtain

$$500 \geqslant \frac{1}{Z}\sqrt{30{,}900^2+18{,}000^2},$$

whence

$$Z \geqslant \frac{\sqrt{30{,}900^2+18{,}000^2}}{500}=71.6 \text{ cm}^3$$

or

$$Z=0.1d^3, \qquad d=\sqrt[3]{\frac{71.6}{0.1}}=8.95 \cong 9 \text{ cm}.$$

Example 81. A transmission shaft (Fig. 186*a*) receives a torque through pulley *I* of a horizontal belt drive and transmits it further by means of pulley *II* of a vertical belt drive. Determin the diameter of the shaft if the tensions in the tight parts of belts which pass over pulleys *I* and *II* are, respectively, 600 and 300 kgf, the tensions in the loose parts are 300 and 150 kgf; the allowable

stress for the shaft material is $[\sigma] = 600$ kgf/cm². Neglect the weight of the pulleys.

Solution. Transferring the forces of tension in the belts to the axis of the shaft, we find that the axis of the shaft is acted on

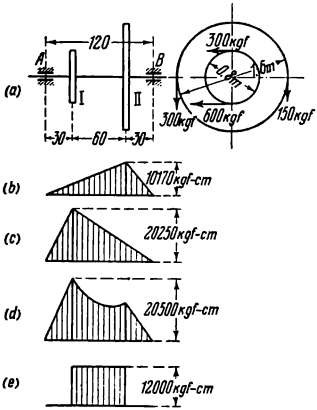


Fig. 186

by a horizontal force $P_h = 900$ kgf in the plane of pulley I and a vertical force $P_v = 450$ kgf in the plane of pulley II. Besides, the middle portion of the shaft is acted on by the torque

$$M_t = 600 \times 40 - 300 \times 40 = 12{,}000 \text{ kgf-cm}.$$

To determine the dangerous section we plot a bending moment diagram due to the forces acting in the vertical plane, a bending moment diagram due to the forces acting in the horizontal plane, and the total moment diagram.

(a) *Construction of Moment Diagram in the Vertical Plane* (Fig. 186*b*). The reactions at the supports are determined from the equations $\Sigma M_A = 0$ and $\Sigma M_B = 0$

$$B_v \times 120 - 450 \times 90 = 0,$$

$$B_v = 450 \frac{90}{120} = 337 \text{ kgf};$$

$$A_v \times 120 - 450 \times 30 = 0,$$

$$A_v = 450 \frac{30}{120} = 113 \text{ kgf}.$$

The moments over the supports are zero; the moment under the force is

$$M_v = A_v \times 90 = 113 \times 90 = 10,170 \text{ kgf-cm}.$$

(b) *Construction of Moment Diagram in the Horizontal Plane* (Fig. 186*c*).

The reactions at the supports are

$$\Sigma M_A = 0, \qquad B_h \times 120 - 900 \times 30 = 0,$$

$$B_h = 900 \frac{30}{120} = 225 \text{ kgf};$$

$$\Sigma M_B = 0, \qquad A_h \times 120 - 900 \times 90 = 0,$$

$$A_h = 900 \frac{90}{120} = 675 \text{ kgf}.$$

The moments over the supports are zero; the moment under the force is

$$M_h = A_h \times 30 = 675 \times 30 = 20,250 \text{ kgf-cm}.$$

(c) *Construction of Total Moment Diagram* (Fig. 186*d*). Knowing the moment M_v due to the vertical forces and the moment M_h due to the horizontal forces for any given section, it is easy to determine the total bending moment for this section from the formula

$$M = \sqrt{M_v^2 + M_h^2}.$$

Figure 186*d* shows the total moment diagram. The total moments at different sections lie, in general, in different planes passing through the shaft axis. Since only the absolute values of the total moments are important in the design, the total moment diagram is plotted conventionally as if all the moments lay in the same plane.

The torque acts only in the middle portion of the shaft, the moment diagram is a straight line parallel to the axis (Fig. 186*e*).

From inspection of the total bending and twisting moment diagrams it is seen that the most highly stressed section is the one coinciding with the middle plane of the left-hand pulley. At this section

$$M = 20,500 \text{ kgf-cm}, \qquad M_t = 12,000 \text{ kgf-cm}.$$

Substituting the values of M, M_t and $[\sigma]$ in formula (12.16), we obtain

$$600 \geqslant \frac{1}{Z} \sqrt{20,500^2 + 12,000^2},$$

whence

$$Z \geqslant \frac{\sqrt{20,500^2 + 12,000^2}}{600} = 39.6 \text{ cm}^3.$$

Since $Z=0.1d^3$, we have

$$d=\sqrt[3]{\frac{Z}{0.1}}=\sqrt[3]{\frac{39.6}{0.1}}=7.34 \text{ cm}.$$

According to the USSR State Standard OST the nearest larger diameter of the shaft is 7.5 cm, which value is taken as the required one.

88. Combined Torsion and Tension or Compression

Combined torsion and tension or compression is encountered, for example, in the design of screws and bolts. The stress distribution around a point on the surface of a twisted and stretched or compressed shaft differs in no way from the stress distribution in the case of torsion combined with bending since bending involves normal tensile and compressive stresses. Therefore, the design formulas (12.18) and (12.23)

$$\sqrt{\sigma^2+4\tau^2}\leqslant[\sigma], \qquad \sqrt{\sigma^2+3\tau^2}\leqslant[\sigma]$$

hold true also for the case of combined torsion and tension or compression. If a round rod is subjected to a tensile or compressive force P and a torsional moment M_t, then

$$\sigma=\frac{P}{A}, \qquad \tau=\frac{M_t}{Z_p}.$$

Formula (12.18) is rewritten as

$$\sqrt{\left(\frac{P}{A}\right)^2+4\left(\frac{M_t}{Z_p}\right)^2}\leqslant[\sigma] \tag{12.25}$$

and formula (12.23) becomes

$$\sqrt{\left(\frac{P}{A}\right)^2+3\left(\frac{M_t}{Z_p}\right)^2}\leqslant[\sigma]. \tag{12.26}$$

Example 82. Determine the diameter d of a shaft stretched by a force $P=9{,}000$ kgf if the torque transmitted by the shaft is $M_t=6{,}000$ kgf-cm. The allowable stress is $[\sigma]=600$ kgf/cm^2.

Solution. Substitute the given data in the design formula (12.25)

$$600\geqslant\sqrt{\left(\frac{9{,}000}{\frac{\pi}{4}d^2}\right)^2+4\left(\frac{6{,}000}{\frac{\pi}{16}d^3}\right)^2}.$$

Square the left-hand and right-hand sides of the above expression

$$36\times10^4\geqslant\frac{81\times10^6}{\frac{\pi^2}{16}d^4}+\frac{4\times36\times10^6}{\frac{\pi^2}{256}d^6}.$$

We take $\pi^2 \cong 10$ and get rid of the denominators

$$36 \times 10^6 d^6 = 16 \times 81 \times 10^6 d^2 + 256 \times 4 \times 36 \times 10^6$$

or

$$d^6 - 360d^2 - 10{,}240 = 0.$$

Denote $d^2 = x$; we then obtain the equation

$$x^3 - 360x - 10{,}240 = 0.$$

We solve this third-degree equation by the method of trial and error. Put $x = 25$; we then obtain

$$15{,}625 - 9{,}000 - 10{,}240 \neq 0.$$

Put $x = 27$

$$19{,}680 - 9{,}720 - 10{,}240 \neq 0.$$

Put $x = 27.2$

$$20{,}120 - 9{,}790 - 10{,}240 \cong 0.$$

Thus, we can take $x = 27.2$; then $d = \sqrt{27.2} = 5.22$ cm.

89. Check Questions

In what case is bending called oblique? How is the design formula constructed?

How is the neutral line inclined with respect to one of the principal axes of the cross section in oblique bending?

Does the neutral line pass through the centroid of the cross section in oblique bending?

What deformations are produced in a beam by a force acting at an angle to its axis?

What is eccentric compression?

How is the maximum stress at a section determined in the general case of eccentric compression or tension?

How is the radius of gyration of a section determined?

When is it necessary to determine the core of the section?

How is the strength condition expressed for combined bending and torsion?

Chapter XIII

Buckling

90. Concept of Buckling

In the types of deformation discussed so far the magnitude of deformation depends linearly on the load. As the load increases gradually, the deformation increases without any abrupt changes and the type of state of stress remains always the same. There are certain cases, however, where the configuration of equilibrium of

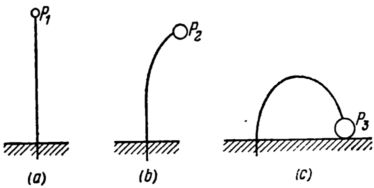


Fig. 187

a body and the state of stress change abruptly as the load increases gradually, and this may result in sudden failure. If a bar is compressed by longitudinal forces until the compressive forces exceed a certain limiting value depending on the length of the bar and its stiffness, the bar undergoes ordinary compression and its axis remains straight. If, however, the compressive forces become larger than this limiting value, the bar buckles suddenly and its axis deflects.

The deflection of the bar brings about a bending moment which produces additional stresses, and the bar may fail suddenly. *The failure of long bars compressed by longitudinal forces is called buckling.*

Consider this phenomenon in greater detail. Take a thin long steel bar (Fig. 187*a*). Assume that the lower end is rigidly fixed and the upper end is subjected to a compressive force acting precisely along the axis of the bar. Initially the bar is only compressed and its axis remains straight under the action of this longitudinal force. If the bar is displaced from its straight-line position by

applying a transverse force, the internal elastic forces bring the bar back to its original straight-line position after this force is removed. This will continue until the compressive force becomes larger than a certain limiting value P_1. In this first stage of buckling the straight-line configuration of equilibrium of the bar is stable. Here we have a complete analogy with the stable equilibrium treated in mechanics.

If the compressive force becomes larger than the limiting value P_1, two configurations of elastic equilibrium of the bar will be possible. The first, straight-line, configuration is now unstable, and the second, deflected, is stable. This means that if the bar receives a lateral deflection for some reason, it will not return to its straight-line configuration of equilibrium but will remain deflected.

In this second stage of buckling even a small increase of the force leads to a considerable deflection of the bar (Fig. 187*b*).

If the force P_2 is further increased, say, to a value P_3, the internal forces will no longer balance the external load and in consequence the bar will develop an additional large deflection. The bar will then break if it is made of a brittle material, or its upper end will bear up against the supporting surface, as is shown in Fig. 187*c*, if the material is ductile.

The limiting value of the force at which the straight-line configuration of equilibrium changes from stable to unstable is called the *critical force*. If the load is smaller than the critical value, the straight-line configuration of stable equilibrium is the only possible one. In the previously studied types of deformation it was assumed that there was a single stable configuration of equilibrium. In cases where a structure may have more than one configurations of equilibrium, in addition to the basic requirements listed in Sec. 1, one more requirement is placed upon it, namely that it maintains its stable configuration of equilibrium.

This is a very important consideration. In practice grave accidents (failures of large railway bridges and other engineering structures) resulted from the loss of stability of one of the structural elements. Failures due to buckling are particularly dangerous since they usually occur suddenly.

When the value of the critical force is exceeded, the bending moment due to the longitudinal forces grows more rapidly, as a consequence of an increase in arm, than the moment of the internal forces. Therefore, the critical force is sometimes identified with the failure load. Compression members must, of course, be so dimensioned that the critical force would be considerably larger than the force actually applied to the member.

If the critical force is denoted by P_{cr} and the allowable force by P_a, the ratio $P_{cr}/P_a = k > 1$ is called the stability factor. Just as the safety factor, the stability factor is taken higher for less

homogeneous materials. Thus, the stability factor is assumed to be about 2.5 and higher for wooden structures, 5 to 6 for cast iron structures, and 1.8 to 3 for steel structures.

91. Euler's Formulas

To design bars for buckling strength one must be able to determine the value of the critical force. The formula for determining this force was first derived by the famous mathematician L. Euler, member of the Petersburg Academy of Sciences. The magnitude of the critical force depends on the end conditions of a bar. Below we consider the determination of the critical force for different end conditions of a bar.

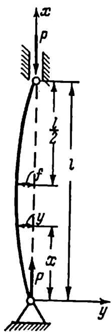


Fig. 188

Case 1 (fundamental). *A bar with hinged ends* (Fig. 188). Under the action of a compressive force equal to or somewhat greater than the critical force, the bar deflects. The bending moment at any section of the bar is

$$M = -Py, \tag{a}$$

i. e., the load (bending moment) itself depends on the deformation (deflection) of the bar. This makes the problem of determining the critical force basically different from all the problems considered thus far.

To derive this formula we assume that the deflected axis or the elastic curve of the bar represents a sine curve.

Denote the magnitude of the deflection at the middle of the bar by f; the elastic curve equation is then

$$y = f \sin \frac{\pi}{l} x. \tag{13.1}$$

There is no deflection, $y=0$, at the ends of the bar, $x=0$ and $x=l$. At the middle of the bar, i. e., at $x=l/2$, the deflection is equal to f, as can easily be seen from Eq. (13.1).

Substituting the value of y from Eq. (13.1) in the expression for the bending moment (a), we obtain

$$M = -Pf \sin \frac{\pi}{l} x. \tag{b}$$

Substituting this expression for the moment in the general equation of the elastic curve (10.3)

$$EI \frac{d^2y}{dx^2} = M, \tag{10.3}$$

we obtain

$$EI\frac{d^2y}{dx^2} = -Pf\sin\frac{\pi}{l}x.$$

We now integrate this expression twice

$$EI\frac{dy}{dx} = Pf\frac{l}{\pi}\cos\frac{\pi}{l}x + C, \tag{c}$$

$$EIy = Pf\frac{l^2}{\pi^2}\sin\frac{\pi}{l}x + Cx + D. \tag{d}$$

Find the constants of integration C and D. The tangent to the elastic curve at the middle of the bar is parallel to the x axis, therefore the derivative dy/dx must be zero at $x = l/2$.

From the equation (c) we then obtain

$$C = 0.$$

At $x = 0$, the deflection $y = 0$; from the equation (d) we have

$$D = 0.$$

The equation (d) can now be written as

$$EIy = Pf\frac{l^2}{\pi^2}\sin\frac{\pi}{l}x.$$

Since $y = f$ at $x = l/2$, we have

$$EIf = P_{cr}f\frac{l^2}{\pi^2}$$

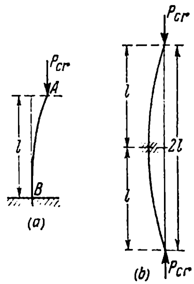


Fig. 189

or

$$P_{cr} = \frac{\pi^2 EI}{l^2}. \tag{13.2}$$

This expression defines the value of the critical force for a bar with hinged ends and is known as *Euler's formula*.

Case 2. *A bar with one end fixed and the other free* (Fig. 189*a*). The critical force for this bar can be found by comparison with the bar of the first case. Indeed, if the axial line of the bar is extended as shown in Fig. 189*b*, it is readily seen that the bar fixed at one end and free at the other is under the same conditions as one-half of the bar hinged at both ends but having a length twice as large. Consequently, to obtain the critical force for a bar with one end fixed and the other free we must substitute $2l$ for l in formula (13.2) for the first case. After making this substitution we obtain

$$P_{cr} = \frac{\pi^2 EI}{(2l)^2}. \tag{13.3}$$

Case 3. *A bar with both ends fixed* (Fig. 190). In this case the ends are assumed to be rigidly fixed, i. e., the ends have no motion at all and the tangents to the elastic curve at the ends of the bar coincide with its axis.

The elastic curve of this bar is made up of four equal parts; each of these parts of length $l/4$ is under the same conditions as the bar fixed at one end (case 2).

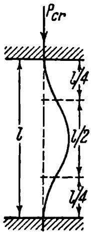


Fig. 190

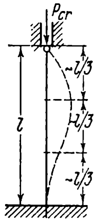


Fig. 191

Therefore, the critical force for the fixed-ended bar can be determined from formula (13.3) by substituting $l/4$ for l. Upon substitution we obtain

$$P_{cr} = \frac{\pi^2 EI}{\left(2\frac{l}{4}\right)^2} = \frac{4\pi^2 EI}{l^2}. \tag{13.4}$$

Consequently, the critical force is 4 times as large as for the hinged bar.

In practice rigid fixing is difficult to achieve. If the slightest amount of rotation occurs at the ends of the bar, the critical force will be much smaller than that determined from formula (13.4). In design practice, therefore, when there is no certainty that the fixing is rigid, the ends of a bar are often assumed to be hinged rather than rigidly fixed to make assurance doubly sure.

Case 4. *A bar with one end fixed and the other hinged* (Fig. 191). The elastic curve of this bar has a point of inflection at a distance from the fixed end of approximately 1/3 the length of the bar. At about the same distance from the hinged end the tangent to the elastic curve is parallel to the axis of the bar.

Thus, with some approximation this bar may be regarded as consisting of three separate bars in which one end is fixed and the other free (case 2). The critical force for each of these parts and consequently for the whole bar can be determined by substituting

$l/3$ for l in formula (13.3). Upon substitution we obtain

$$P_{cr}=\frac{\pi^2EI}{\left(2\frac{l}{3}\right)^2}\cong\frac{2\pi^2EI}{l^2}. \tag{13.5}$$

Formulas (13.2) to (13.5) for determining the critical forces in the above four cases of end conditions may be combined into one formula

$$P_{cr}=\frac{\pi^2EI}{(\mu l)^2}. \tag{13.6}$$

The value of the quantity μ appearing in the denominator, called the *length reduction factor*, is easily determined for the four cases of end conditions from comparison of the relevant formulas with the general formula (13.2). We obtain

for case 1, $\mu=1$;
for case 2, $\mu=2$;
for case 3, $\mu=1/2$;
for case 4, $\mu=2/3$.

The product of the actual length l of the bar and the length reduction factor μ is called the reduced or effective length of the bar. Denoting the effective (design) length of the bar by l_{eff}

$$l_{eff}=\mu l, \tag{13.7}$$

we rewrite the general formula (13.6) as follows

$$P_{cr}=\frac{\pi^2EI}{l_{eff}^2}. \tag{13.8}$$

Thus, the critical force for all cases of end conditions can be determined from formula (13.8). It should be remembered, however, that in this formula l_{eff} represents the design, or effective, length and not the actual length of the bar.

The concept of the effective length was first introduced by the Professor of the Petersburg Institute of Railway Transport F. Yasinsky in 1892.

It should be noted that one must substitute the minimum value of the axial centroidal moment of inertia of the section in the formula for the critical force (if the axial centroidal moments of inertia of the section are not the same) since *the bar always tends to bend in the plane of least rigidity*.

Introducing the stability factor into formula (13.8), we obtain the general formula for determining the allowable force in the case of buckling

$$P_a=\frac{P_{cr}}{k}=\frac{\pi^2EI}{kl_{eff}^2}. \tag{13.9}$$

In the following discussion, formulas (13.8) and (13.9) will be referred to as Euler's formulas, as is customary.

Denoting the cross-sectional area by A, we can easily derive an expression for determining the critical stress corresponding to the critical force and the allowable stress corresponding to the allowable force

$$\sigma_{cr}=\frac{P_{cr}}{A}=\frac{\pi^2 EI}{Al_{eff}^2}, \tag{13.10}$$

$$\sigma_a=\frac{P_a}{A}, \tag{13.11}$$

where σ_a is the allowable stress in buckling. Formula (13.10) is usually transformed by introducing the radius of gyration of the section. Since

$$I=Ai^2,$$

we have

$$\sigma_{cr}=\frac{\pi^2 EAi^2}{Al_{eff}^2}=\frac{\pi^2 E}{\left(\frac{l_{eff}}{i}\right)^2}=\frac{\pi^2 E}{\lambda^2}. \tag{13.12}$$

The quotient of l_{eff} and i is generally called the *slenderness ratio* and denoted by λ. From formula (13.12) it follows that the critical stress in the bar in the case of buckling is inversely proportional to the square of the ratio of the design length to the radius of gyration.

92. Limitations of Euler's Formula. Table for Column Design

In the derivation of his critical-force formula for compression members L. Euler assumed that the material was sufficiently elastic and followed Hooke's law.

As is known, the material follows Hooke's law only until the stress is below the proportional limit. Consequently, Euler's formula for different materials must also have certain distinct limits of applicability. It is only valid until the critical stress in the bar exceeds the proportional limit of the material. In short members the critical stress defined by Euler's formula is higher than the proportional limit. *For short members*, therefore, *Euler's formula is completely inapplicable.*

The limit of applicability of Euler's formula is the case when the critical stress is equal to the proportional limit. On this basis, for any material it is possible to determine the limiting values of the geometrical proportions of a column for Euler's formula to apply.

Substituting the proportional limit (σ_p) for the critical stress (σ_{cr}) in formula (13.12) and determining the slenderness ratio λ therefrom,

we obtain

$$\lambda \geqslant \sqrt{\frac{\pi^2 E}{\sigma_p}}. \tag{13.13}$$

Determine, for example, the limit of applicability of Euler's formula for steel of grade 30 having a modulus of elasticity $E = 2.0 \times \times 10^6$ kgf/cm^2 and a proportional limit $\sigma_p = 2{,}000$ kgf/cm^2.

Substituting the values of E and σ_p in formula (13.13), we have

$$\lambda = \sqrt{\frac{\pi^2 \times 2.0 \times 10^6}{2{,}000}} = 100.$$

If the slenderness ratio λ for a column of this steel is less than 100, it is apparent that the critical stress is higher than the proportional limit and Euler's formula (13.12) is applicable in this case only if the modulus E is replaced by E_R depending on the value of σ_{cr}.

In a similar manner we can calculate the limit of applicability of Euler's formula for any other material substituting the values of the modulus of elasticity and the proportional limit of the given material in Formula (13.13). For cast iron, Euler's formula is applicable for a slenderness ratio $\lambda \geqslant 80$, for pine—for $\lambda \geqslant 100$. The so-called reduced, or effective modulus $E_R < E$ is found by experiment.

In cases where the slenderness ratio of a member is less than the limiting value for the given material, i. e., where the critical stress as determined from Euler's formula is higher than the proportional limit, empirical formulas have been proposed for determining the critical stress.

Having collected and analysed a great body of experimental data, F. Yasinsky showed that the critical stresses for members of small slenderness ratio (for which Euler's formula does not apply) can be determined from the equation

$$\sigma_{cr} = a - b\lambda, \tag{3.14}$$

where a and b are quantities depending on the type of material. The values of these coefficients for various materials are given in handbooks.

The allowable stress σ_a in the case of buckling within and beyond the proportional limit depends on the material and the slenderness ratio λ of a member and it may be regarded as a certain fraction φ of the allowable stress $[\sigma]$ in simple compression, i.e.,

$$\sigma_a = \varphi [\sigma]. \tag{3.15}$$

The factor φ is always less than unity. It is called the *allowable stress reduction factor for compression members*. The factor φ depends on the material and the slenderness ratio λ of a column. Thus,

introducing the reduction factor φ, a check on the buckling strength of a column can be made as in the case of simple compression but with a reduced allowable value of the compressive stress.

Table 10 gives factors φ for basic structural materials in the case of hinged columns.

As stated above, for columns with other than hinged end conditions it is necessary to take the effective length in determining the slenderness ratio.

For intermediate values of the slenderness ratio which are not indicated in Table 10, the factors φ are determined by linear interpolation.

Table 10. Factors φ

Slenderness ratio, λ	Values				
	Steel of grade 40, 30, 20	Steel of grade 50	Steel $\sigma_y > 3{,}200$ kgf/cm²	Cast Iron	Wood
0	1.00	1.00	1.00	1.00	1.00
10	0.99	0.98	0.97	0.97	0.99
20	0.96	0.95	0.95	0.91	0.97
30	0.94	0.92	0.91	0.81	0.93
40	0.92	0.89	0.87	0.69	0.87
50	0.89	0.86	0.83	0.57	0.80
60	0.86	0.82	0.79	0.44	0.71
70	0.81	0.76	0.72	0.34	0.60
80	0.75	0.70	0.65	0.26	0.48
90	0.69	0.62	0.55	0.20	0.38
100	0.60	0.51	0.43	0.16	0.31
110	0.52	0.43	0.35	—	0.25
120	0.45	0.37	0.30	—	0.22
130	0.40	0.33	0.26	—	0.18
140	0.36	0.29	0.23	—	0.16
150	0.32	0.26	0.21	—	0.14
160	0.29	0.24	0.19	—	0.12
170	0.26	0.21	0.17	—	0.11
180	0.23	0.19	0.15	—	0.1
190	0.21	0.17	0.14	—	0.09
200	0.19	0.16	0.13	—	0.08

93. Examples of Design for Buckling Strength

Problems dealing with the buckling strength of structural members may be divided into two types: the determination of allowable loads and the selection of sections.

(a) *Determination of Allowable Loads.*

Example 83. Determine the allowable load for a hollow cast iron column of length $l=6$ m; the outer diameter of the section is $D=20$ cm, the inner diameter $d=18$ cm, $E=10^6$ kgf/cm². Assume both ends hinged. The stability factor is $k=5$.

Solution. The moment of inertia of the column section is

$$I=\frac{\pi}{64}D^4-\frac{\pi}{64}d^4=\frac{\pi}{64}20^4-\frac{\pi}{64}18^4=7{,}854-5{,}153=2{,}701\ \text{cm}^4.$$

The cross-sectional area of the column is

$$A=\frac{\pi}{4}D^2-\frac{\pi}{4}d^2=\frac{\pi}{4}20^2-\frac{\pi}{4}18^2=314-254=60\ \text{cm}^2.$$

The radius of gyration of the section is

$$i=\sqrt{\frac{I}{A}}=\sqrt{\frac{2{,}701}{60}}=6.71\ \text{cm}.$$

The design length of the column (case 1, $\mu=1$) is

$$l_{eff}=\mu l=1\times 6=6\ \text{m}.$$

The slenderness ratio of the column is

$$\lambda=\frac{l_{eff}}{i}=\frac{600}{6.71}\cong 90>80.$$

Consequently, the given column can be designed by Euler's formula (13.9)

$$P_a=\frac{\pi^2 E I_{\min}}{k l_{\text{eff}}^2}=\frac{\pi^2\times 10^6\times 2{,}701}{5\times 600^2}=14{,}800\ \text{kgf}.$$

Example 84. Determine the allowable compressive force for a 30 steel member of length $l=30$ cm and diameter $d=1$ cm if the upper end of the member is hinged and the lower end is fixed.

Solution. The radius of gyration of the circular section is

$$i=\sqrt{\frac{I}{A}}=\sqrt{\frac{\pi d^4\times 4}{64\pi d^2}}=\frac{d}{4}=\frac{1}{4}=0.25\ \text{cm}.$$

The design length of the member (case 4, $\mu=2/3$) is

$$l_{\text{eff}}=\mu l=\frac{2}{3}\times 30=20\ \text{cm}.$$

The slenderness ratio of the member is

$$\lambda=\frac{l_{\text{eff}}}{i}=\frac{20}{0.25}=80<100.$$

Consequently, the allowable force for this member cannot be determined by Euler's formula. We determine it using the table of factors φ.

For the given slenderness ratio $\lambda = 80$, the reduction factor $\varphi = 0.75$. The allowable stress for 30 steel in simple compression is taken as $[\sigma] = 1{,}600$ kgf/cm²; the allowable stress in buckling is then

$$\sigma_a = \varphi[\sigma] = 0.75 \times 1{,}600 = 1{,}200 \text{ kgf/cm}^2.$$

The allowable force is

$$P_a = \sigma_a A = \sigma_a \frac{\pi d^2}{4} = 1{,}200 \times 0.785 \times 1 = 942 \text{ kgf}.$$

(b) *Selection of Sections.* Since buckling always occurs in the plane of least rigidity, it is more advantageous from the standpoint of saving of material to choose sections with equal moments of inertia with respect to the centroidal axes, i. e., such that $I_x = I_y$.

From Table 10 it is seen that the allowable stress in the case of buckling depends on the slenderness ratio of a column, i. e., on the ratio $l_{eff}/i = \lambda$; the smaller the ratio, the larger is the allowable stress. Consequently, for a given length of a column and a given cross-sectional area it is more advantageous to choose a section in which the material is distributed as far from the principal centroidal axes as possible. Therefore, an annular section is much more advantageous in this respect than a solid circular section.

Thus, for example, a solid section of diameter $d = 9$ cm has an area $A = 63.6$ cm² and a radius of gyration $i = 2.25$ cm. An annular section of nearly the same area, $A = 62.9$ cm², of outer diameter 21 cm and wall thickness 1 cm has a radius of gyration

$$i = \sqrt{\frac{I}{A}} = \sqrt{\frac{3{,}150}{62.9}} = 7.07 \text{ cm}$$

which is more than three times the radius of gyration of the solid section. For one and the same length of a column, say, $l = 4$ m, the reduction factor for the solid section is $\varphi = 0.243$, and for the annular section $\varphi = 0.815$. Consequently, the allowable stress for the annular section is about 3.3 times that for the solid section.

After the shape of the section is chosen, its dimensions are determined by the method of trial and error. Using Euler's formula and assigning a stability factor k, determine the moment of inertia I

$$I = \frac{P_a k l_{eff}^2}{\pi^2 E}.$$

From this value of I and tables on standard shapes select a section and determine the minimum radius of gyration i. Further find the slenderness ratio λ of the column; if the slenderness ratio is not less than the limiting value for the given material, this completes the choice of the section. Otherwise, for the slenderness ratio obtained take the reduction factor φ from Table 10 and find the corresponding allowable stress. If this allowable stress is less than

that for the chosen section, the section is increased by eye. For the increased section, determine again the radius of gyration and then the slenderness ratio. From the new value of the slenderness ratio determine the allowable stress. If the allowable stress differs only slightly from the stress on the section of new dimensions, this completes the choice of the section. Otherwise it is necessary again to change the section, increasing it if the allowable stress is lower than the stress obtained or decreasing it if the allowable stress is higher than the resulting stress acting on the section.

In the foregoing method, in choosing cross-sectional dimensions we assign a stability factor and determine the section in the first trial from Euler's formula.

It is possible, however, to assign a factor φ as a first approximation, and not a stability factor. This procedure as well as the first one will be illustrated by the following examples.

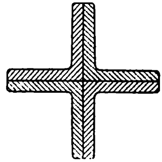


Fig. 192

Example 85. Choose the section of a truss member made up of four equal angles (Fig. 192) if the compressive force is $P = 23{,}000$ kgf, the length of the member is $l = 3$ m. Both ends of the bar are fixed. The allowable stress in compression $[\sigma]$ is taken as 1,000 kgf/cm².

Solution. Determine first the angle number from Euler's formula adopting a stability factor $k = 5$. For the case of both ends of the bar fixed, $\mu = 0.5$; therefore, its design length is

$$l_{\text{eff}} = \mu l = 3 \times 0.5 = 1.5 \text{ m}.$$

From formula (13.9) we obtain the value for the moment of inertia as

$$I = \frac{P_a k l_{\text{eff}}^2}{\pi^2 E} = \frac{23{,}000 \times 5 \times 150^2}{9.87 \times 2 \times 10^6} = 131 \text{ cm}^4.$$

Consequently, the moment of inertia of one equal angle with respect to its base must be

$$I_{\llcorner} = 131 : 4 \cong 32.8 \text{ cm}^4.$$

The nearest angle number from the table on rolled steel sections of GOST 8509-57 is No. 7 with a wall thickness of 5 mm; it has a moment of inertia $I_{\llcorner} = 31.9$ cm⁴ and an area $A_{\llcorner} = 6.86$ cm².

Determine the radius of gyration of the bar section

$$i = \sqrt{\frac{4I_{\llcorner}}{4A_{\llcorner}}} = \sqrt{\frac{31.9}{6.86}} = 2.16 \text{ cm}.$$

The slenderness ratio of the bar is

$$\lambda = \frac{l_{\text{eff}}}{i} = \frac{150}{2.16} = 69 < 100.$$

Consequently, Euler's formula is not applicable. Take a reduction factor φ from Table 10 for the above slenderness ratio. For $\lambda = 69$ the factor is

$$\varphi = 0.86 - \frac{(0.86 - 0.81)\,9}{10} = 0.815.$$

The allowable stress is

$$\sigma_a = [\sigma]\,\varphi = 1{,}000 \times 0.815 = 815 \text{ kgf/cm}^2.$$

Check the resulting stress in the bar if the angle is No. 7

$$\sigma = \frac{23{,}000}{4 \times 6.86} = 838 \text{ kgf/cm}^2 > 815 \text{ kgf/cm}^2.$$

Consequently, the section of the bar should be increased. Take a No. 7 angle of wall thickness 6 mm; it has a moment of inertia $I^{\llcorner} = 37.6$ cm^4 and an area $A^{\llcorner} = 8.15$ cm^2. The radius of gyration of the section is

$$i = \sqrt{\frac{4I^{\llcorner}}{4A^{\llcorner}}} = \sqrt{\frac{37.6}{8.15}} = 2.15 \text{ cm}.$$

The slenderness ratio is

$$\lambda = \frac{l_{\text{eff}}}{i} = \frac{150}{2.15} = 70.$$

The corresponding reduction factor is $\varphi = 0.81$. The allowable stress is

$$\sigma_a = 1{,}000 \times 0.81 = 810 \text{ kgf/cm}^2.$$

Check the resulting stress in the column

$$\sigma = \frac{23{,}000}{4 \times 815} = 736 \text{ kgf/cm}^2 < 810 \text{ kgf/cm}^2.$$

This stress is lower than the allowable value, therefore we adopt the No. 7 section of wall thickness 6 mm.

Example 86. Determine the diameter of a steel bar (steel of grade 50) of length $l = 70$ cm subjected to a compressive force of 12 tons if the ends of the bar are hinged. The allowable stress in simple compression is taken as $[\sigma] = 1{,}500$ kgf/cm^2.

Solution. As a first approximation we assume $\varphi = 0.5$; the required cross-sectional area is then

$$A \geqslant \frac{P}{\varphi\,[\sigma]} = \frac{12{,}000}{0.5 \times 1{,}500} = 16 \text{ cm}^2.$$

The radius of the bar section is

$$r=\sqrt{\frac{A}{\pi}}=\sqrt{\frac{16}{3.14}}=2.26\text{ cm}.$$

The radius of gyration of the section is

$$i=\frac{r}{2}=\frac{2.26}{2}=1.13\text{ cm}.$$

The slenderness ratio of the bar is

$$\lambda=\frac{l_{\text{eff}}}{i}=\frac{70}{1.13}=62.$$

For 50 steel, to the slenderness ratio $\lambda=60$ corresponds the factor $\varphi=0.82$ and to the slenderness ratio $\lambda=70$ corresponds the factor $\varphi=0.76$. Consequently, to the slenderness ratio $\lambda=62$ corresponds the factor

$$\varphi=0.82-\frac{(0.82-0.76)\,2}{10}=0.81.$$

For this value of φ, the allowable stress in buckling is

$$\sigma_a=\varphi\,[\sigma]=0.81\times 1{,}500=1{,}215\text{ kgf/cm}^2.$$

Check the stress induced in the bar if its cross-sectional area is $A=16$ cm^2

$$\sigma=\frac{12{,}000}{16}=750\text{ kgf/cm}^2<1{,}215\text{ kgf/cm}^2,$$

i. e., it is much lower than the allowable stress (1,215 kgf/cm^2), and hence the cross-sectional area should be decreased. Let us try $A=10$ cm^2. The radius of gyration of the section is

$$i=\frac{r}{2}=\frac{\sqrt{\frac{10}{3.14}}}{2}=0.89\text{ cm}.$$

The slenderness ratio of the bar is

$$\lambda=\frac{l_{\text{eff}}}{i}=\frac{70}{0.89}=79.$$

The corresponding value of the factor φ is, by interpolation.

$$\varphi=0.76-\frac{(0.76-0.70)\,9}{10}=0.71.$$

The allowable stress in buckling is

$$\sigma_a=\varphi\,[\sigma]=0.71\times 1{,}500=1{,}065\text{ kgf/cm}^2.$$

If the cross-sectional area of the bar is $A = 10$ cm², the stress is

$$\sigma = \frac{12{,}000}{10} = 1{,}200 \text{ kgf/cm}^2 > 1{,}065 \text{ kgf/cm}^2.$$

In this case the bar is overstressed, and therefore the cross-sectional area should be somewhat increased.

Take $A = 11$ cm². In this case i, λ and φ are, respectively,

$$i = \frac{1}{2}\sqrt{\frac{11}{3.14}} = 0.93, \qquad \lambda = \frac{70}{0.93} = 75, \quad \varphi = 0.73.$$

The allowable stress in buckling is

$$\sigma_a = 0.73 \times 1{,}500 = 1{,}095 \text{ kgf/cm}^2.$$

Check the stress in the bar

$$\sigma = \frac{12{,}000}{11} = 1{,}090 \text{ kgf/cm}^2.$$

This stress differs only slightly from the allowable value σ_a; consequently, the cross-sectional area $A = 11$ cm² satisfies the stability condition.

In the above discussion, only the fundamental concepts of buckling of compression members were considered. In practice we encounter much more complicated cases of instability of both compression members and other elements having one dimension considerably smaller than its other dimensions, such as thin-walled beams, tubes, thin plates. The analysis of these cases of instability is beyond the scope of this book.

94. Check Questions

What is buckling?

What is the critical force?

What is the stability factor?

What stability factors are adopted for wooden, steel and cast iron structures?

Write Euler's formula in the general form.

What is the length reduction factor and what are its values for four common end conditions?

Are intermediate cases possible?

What moment of inertia is substituted in Euler's formula? Why?

In what case does a bar have equal probability of buckling in all directions?

Define the slenderness ratio of a bar. Write its formula.

How is a compression member designed for the buckling strength if the slenderness ratio is such that Euler's formula is inapplicable?

What is the formula for σ_{cr} at stresses greater than elastic stresses?

What is the factor φ incorporated in the allowable compressive stress?

What does the factor φ depend on?

Chapter XIV

Strength under dynamic and repeated loading

95. Concepts of Dynamic and Repeated Loading

In all the problems considered thus far it has been assumed that the imposed loads are static, i.e., that they do not vary in time. But in problems of machine design we usually deal with members having complex motions. Thus, a piston of an engine moves non-uniformly in the cylinder with an acceleration varying in magnitude and direction, and the particles of a circular ring rotating with constant revolutions move with an acceleration constant in magnitude but variable in direction. The particles of a non-uniformly moving member are acted on by inertia forces which can be determined if the mass of a particle and its acceleration are known.

An example of a static load or the static action of a load is provided by a weight suspended by a chain. The action remains static if the weight is lifted with a constant velocity, i.e., with no acceleration. But the same weight when lifted with a certain acceleration will act dynamically on the chain. In the design of the chain we must take into account not only the weight but also the inertia force of the weight in this case. This inertia force may considerably exceed the weight itself.

Among dynamic loads are also impact loads. Examples of impact loads are the action of a falling ram on a driven pile, the action of a hammer on a forged member and an anvil, the explosion of powder in a gun barrel, etc. Furthermore, machine parts may be subjected to impact loads due to the presence of clearances in the connections of parts. In the latter case the impact load is detrimental and precautions should be taken to reduce its effect.

Methods of designing machine parts for impact loads are very complicated and are as yet inadequately developed. Besides dynamic loads, in designing machines and some structures we often have to deal with repeated loads which produce fluctuating stresses varying periodically in time. Thus, the loads acting on the connecting rod and the crankshaft in a piston engine vary continually and are repeated with every revolution (two-stroke engine) or with every two revolutions (four-stroke engine). Methods of designing for repeated loads were first developed rather recently.

Here we shall consider the simplest examples of designing for dynamic loads and in somewhat greater detail methods of designing parts subjected to repeated loads.

96. Design of a Uniformly Rotating Ring

As an example of a member "loaded" by inertia forces we consider a uniformly rotating thin ring. Let r denote the mean radius of the ring, A the cross-sectional area, γ the specific weight of

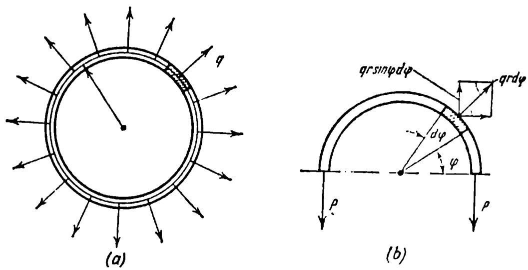


Fig. 193

the material, v the circumferential velocity of the ring per second, g the acceleration of gravity.

The rotating ring is "loaded" by centrifugal inertia forces uniformly distributed along the ring (Fig. 193a). The centrifugal force calculated per unit length of the ring is equal to the product of the mass m of an element of the ring of unit length and the centripetal acceleration v^2/r, i.e.,

$$q = m\frac{v^2}{r};$$

the mass of an element of the ring of unit length may be expressed as

$$m = \frac{A1\gamma}{g}.$$

Consequently,

$$q = \frac{A\gamma}{g}\frac{v^2}{r}. \tag{a}$$

The inertia force of an infinitesimal element of the ring cut out by two planes making a central angle $d\varphi$ (Fig. 193b) is $qrd\varphi$.

Find the tensile force P acting at a section of the ring. To do this, we cut the ring through the diameter and write the equilibrium condition taking the sum of vertical components $qr \sin\varphi\, d\varphi$ of all forces $qr\, d\varphi$ acting on one-half of the ring

$$2P = 2\int_0^{\pi/2} qr \sin\varphi\, d\varphi = 2qr,$$

whence

$$P = qr.$$

Assuming that all the fibres are stretched identically in a thin ring, we find the tensile stress on the section of the ring

$$\sigma = \frac{P}{A} = \frac{qr}{A}$$

or, substituting the value of q from (a), we obtain

$$\sigma = \frac{\gamma}{g} v^2. \tag{14.1}$$

Since

$$v = \frac{2\pi r n}{60},$$

we have

$$\sigma = \frac{\gamma}{g}\left(\frac{\pi r n}{30}\right)^2. \tag{14.2}$$

Determine now the amount the radius of the rotating ring elongates. The unit elongations of the fibres of the ring are

$$\varepsilon = \frac{2\pi(r+\Delta r) - 2\pi r}{2\pi r} = \frac{\Delta r}{r}.$$

On the other hand, the unit elongation is

$$\varepsilon = \frac{\sigma}{E}.$$

Consequently,

$$\frac{\Delta r}{r} = \frac{\sigma}{E},$$

whence

$$\Delta r = \frac{\sigma}{E} r \tag{14.3}$$

or, substituting the value of σ, we have

$$\Delta r = \frac{\gamma}{g}\frac{v^2 r}{E}. \tag{14.4}$$

Example 87. Determine the stress and the elongation of the radius of a flywheel, neglecting the effect of spokes, if the mean radius $r = 1.5$ m, the number of revolutions $n = 100$ rpm, the specific weight of the material $\gamma = 0.0075$ kgf/cm^3, the modulus of elasticity $E = 2\times10^6$ kgf/cm^2. Substituting the numerical values, we find the stress from formula (14.2)

$$\sigma = \frac{0.0075}{981}\left(\frac{3.14\times150\times100}{30}\right)^2 = 1{,}890 \text{ kgf/cm}^2.$$

The increase in the radius of the flywheel is easier to determine from formula (14.3) using the known stress

$$\Delta r = \frac{1{,}890}{2\times10^6}\,150 = 0.142 \text{ cm}.$$

97. The Stress and Strain in a Rod Subjected to Impact Loading

As the simplest example we consider the determination of stress and strain in a rod under axial impact.

Take a bar AB (Fig. 194) having a collar CD at its lower end. Put a weight Q on it and rigidly fix the upper end of the bar. If the weight Q is now raised to a height h and allowed to drop, it will fall on to the collar. Impact is produced and in consequence the bar AB elongates. The elongation is larger than it would be if the weight Q were statically applied to the bar. The resulting elongation of the bar will be called dynamic elongation and designated Δl_d. This elongation can be determined on the assumption that the stress induced in the bar is within the elastic limit.

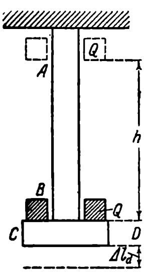


Fig. 194

As the weight Q falls from the height h it does an amount of work equal to Qh before striking the collar of the bar. On coming in contact with the collar, the weight moves down an additional distance Δl_d with the collar due to the extension of the bar. The work done by the weight during its downward motion with the collar is $Q\Delta l_d$.

Thus, the total work done by the weight is

$$Qh + Q\Delta l_d \quad \text{or} \quad Q(h+\Delta l_d).$$

If the mass of the bar and the energy dissipated during the impact are neglected, all this work is expended in stretching the bar. The strain energy in the bar is, according to formula (2.7), $\Delta l_d^2 EA/2l$.

Consequently, to determine Δl_d we can write the equation

$$Q(h+\Delta l_d)=\frac{\Delta l_d^2 EA}{2l}. \tag{a}$$

Opening the parentheses and dividing through by $EA/2l$, we obtain

$$\Delta l_d^2-\frac{2Ql}{EA}\Delta l_d-\frac{2Qhl}{EA}=0. \tag{b}$$

Since the quantity Ql/EA represents, according to formula (2.4), the elongation Δl_{st} of the bar produced by the load Q acting statically, we rewrite the expression (b) as follows

$$\Delta l_d^2-2\Delta l_{st}\,\Delta l_d-2\,\Delta l_{st}h=0,$$

whence

$$\Delta l_d=\Delta l_{st}+\sqrt{\Delta l_{st}^2+2\,\Delta l_{st}h}. \tag{14.5}$$

This formula can also be rewritten as

$$\Delta l_d=\Delta l_{st}k,$$

where k is the impact factor

$$k=1+\sqrt{1+\frac{2h}{\Delta l_{st}}}.$$

Note that the elongation calculated by formula (14.5) occurs in the bar when the load is in its lowest position after impact. The bar starts vibrating in the longitudinal direction due to the impact. The vibrations will gradually be damped out and the bar will come to rest after a certain length of time. At this time the elongation of the bar is such as if the weight Q were statically applied, i.e., the elongation is

$$\Delta l_{st}=\frac{Ql}{EA}.$$

At the time of lowest position of the weight after impact the stress in the bar is, according to Hooke's law,

$$\sigma_d=E\varepsilon_d=E\frac{\Delta l_{st}+\sqrt{\Delta l_{st}^2+2\,\Delta l_{st}h}}{l}. \tag{14.6}$$

If the height h is very large compared with Δl_{st}, we obtain from formula (14.5)

$$\Delta l_d\cong\sqrt{2\,\Delta l_{st}h}. \tag{14.7}$$

The stress is in this case

$$\sigma_d=E\frac{\sqrt{2\,\Delta l_{st}h}}{l}. \tag{14.8}$$

If the weight Q does not fall from a height but is suddenly applied, then $h=0$ and we have from formula (14.5)

$$\Delta l_d = 2\,\Delta l_{st}. \tag{14.9}$$

The stress is then

$$\sigma_d = E\frac{2\,\Delta l_{st}}{l} = 2\sigma_{st}. \tag{14.10}$$

Consequently, *both the elongation and stress due to a load suddenly applied are twice as great as those caused by the same load when statically applied.*

Formula (14.5) for determining the magnitude of the elongation of a bar produced by axial impact is valid not only in the case of axial impact but also for bending produced by impact. If a weight P acts statically at the middle of a simply supported beam of length l, the deflection is expressed by formula (10.13)

$$f_{st} = \frac{Pl^3}{48EI}.$$

If the same weight P falls from a height h on to the middle of the beam, the dynamic deflection is, according to formula (14.5),

$$f_d = f_{st} + \sqrt{f_{st}^2 + 2f_{st}h}. \tag{14.5'}$$

The maximum static stress at mid-length is

$$\sigma_{st} = \frac{M}{Z} = \frac{Pl}{4Z}.$$

The maximum dynamic and static stresses are in the same ratio as the dynamic and static deflections, i.e.,

$$\sigma_d = \sigma_{st}\frac{f_d}{f_{st}} = \frac{Pl}{4Z}\left(\frac{f_{st} + \sqrt{f_{st}^2 + 2f_{st}h}}{f_{st}}\right)$$

or

$$\sigma_d = \frac{Pl}{4Z}\left(1 + \sqrt{1 + \frac{2h}{f_{st}}}\right) = \frac{Pl}{4Z}k, \tag{14.6}$$

where k is the impact factor.

If $h=0$, then $k=2$, i.e., the dynamic stress is twice as great as the static stress, i.e., it is the same as in the case of axial impact. Of course, formula (14.5′) is also valid if the weight P falls on to the beam at any other location; then f_{st} is the static deflection at that location. Likewise, the beam may be supported in any other possible way.

However, not only the stresses and strains produced by impact differ from those caused by a statically applied load, but the material itself behaves under impact loading in a different way than under static loading. As investigations show, the mechanical cha-

racteristics of materials obtained from static tests are inadequate for predicting the behaviour of the material under impact loading. Even high ductility of a material revealed in a static test cannot ensure good resistance of this material to impact loads. Therefore, in designing structural parts which have to withstand impact loads in actual service it is necessary to know the capacity of the material to endure the instantaneous action of a load; this capacity of the material is determined in impact testing.

98. Impact Testing of Metals

The impact test most commonly performed at present is the test using notched specimens and carried out in a pendulum-type impact testing machine. Figure 195 shows a schematic diagram of an

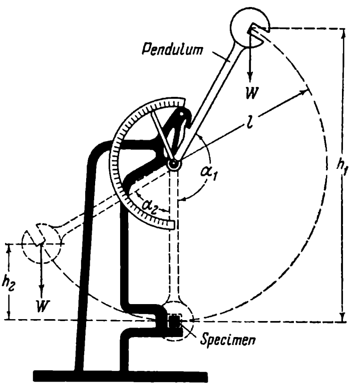


Fig. 195

impact testing machine, and Fig. 196 shows a notched specimen employed in impact testing.

A measure of the resistance of metals to impact loads is the amount of work required to bring a specimen to rupture and calculated per unit area of its section. Thus, if the work used in breaking the specimen is denoted by W and the cross-sectional area of the specimen by A, the quantity

$$a_k = \frac{W}{A} \text{ m-kgf/cm}^2$$

is a measure of impact resistance called the impact strength or the modulus of toughness of the material.

Example 88. Determine the height h (Fig. 194) from which a weight $Q=100$ kgf must be dropped if the stress induced in a steel

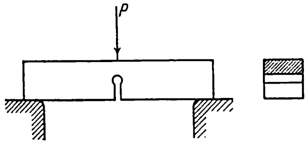


Fig. 196

bar of length $l=1$ m and cross-sectional area $A=1$ cm² is to be equal to the elastic limit $\sigma_e=2{,}000$ kgf/cm²; $E=2\times10^6$ kgf/cm².

Solution. Determine the static elongation

$$\Delta l_{st}=\frac{Ql}{EA}=\frac{100\times100}{2\times10^6\times1}=0.005 \text{ cm}.$$

The height h is determined from formula (14.6) by substituting σ_e for σ_d

$$2{,}000=2\times10^6\,\frac{0.005+\sqrt{0.005^2+2\times0.005h}}{100},$$

whence

$$h=\frac{0.009025-0.000025}{0.01}=\frac{0.009}{0.01}=0.9 \text{ cm}.$$

From this it is seen that a weight falling from a small height may produce stresses many times the static stress.

99. Fatigue of Metals

More than a hundred years ago it was observed that machine and structural parts subjected to fluctuating stresses over a relatively long period of time may fail suddenly without noticeable permanent deformation at stress values much smaller than the ultimate strength of the material. This phenomenon was termed fatigue of materials. The first step to find out the cause of these failures was to see whether the ultimate strength of the material was decreased after fluctuating stresses acted for a long time. Experiments showed, however, that repeatedly applied stresses did not change the mechanical properties of a material. The assumption that fluctuating stresses change the structure of a material and make it brittle was not supported either. This assumption was based on the fact that the material with sufficiently ductile properties fails under

fluctuating stresses as a brittle material without perceptible permanent deformation.

Numerous experiments show that when varying stresses are higher than a certain value for a given material, the material develops a crack after a number of stress cycles. The crack is usually started at the surface of the material at points of maximum stresses and at local defects in the material. Once formed, the crack, initially very fine and invisible to the naked eye, spreads gradually into the interior of the material. The plastic deformation is concentrated only at the crack, therefore no perceptible permanent deformation is revealed after fracture. The parts of the material on the two sides of the crack rub against each other and gradually smooth off the contact surface under repeated loading and unloading.

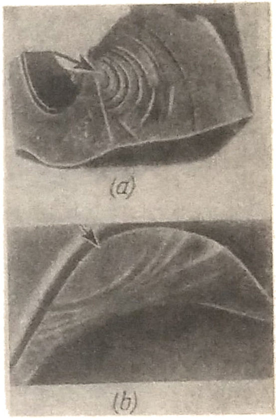


Fig. 197

After the crack attains such a size that the section of the body becomes markedly weakened, a sudden fracture occurs. The resulting fracture surface is always of a brittle type. The fatigue fracture surface has a characteristic structure. Figure 197*a* and *b* shows this kind of fracture. It exhibits clearly two different zones: the first zone with a smooth ground surface and the zone of the final brittle fracture. The point of origin of the initial crack is shown by an arrow. Concentric lines and bands spreading out from this point indicate gradual propagation of the crack.

In the great majority of cases the cause of failure of machine parts is the fatigue of the material, i.e., the sudden fracture under repeated loads at stress values smaller than the ultimate strength. The results of static and impact tests enable one to judge only to some degree the capacity of the material to endure repeatedly applied loads. The determination of this important material characteristic necessary in the design of machines and structures subjected to varying stresses requires special types of test called endurance or fatigue tests.

Before proceeding to the description of fatigue (endurance) tests, we shall discuss some concepts which will be needed in what follows.

Fluctuating stresses in machine parts vary between two extreme values, the maximum stress σ_{max} and the minimum stress σ_{min}. Figure 198 shows a periodic variation of stresses in time. The number of stress cycles per second is called the stress frequency.

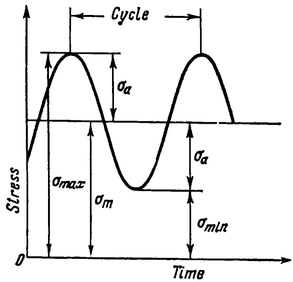


Fig. 198

The algebraic half-sum of the maximum and minimum stresses of the cycle is called the mean stress of the cycle and denoted by σ_m, or τ_m in the case of shearing stresses, i.e.,

$$\sigma_m = \frac{\sigma_{max}+\sigma_{min}}{2},$$

$$\tau_m = \frac{\tau_{max}+\tau_{min}}{2}. \tag{14.11}$$

The absolute value of the algebraic difference between the maximum and minimum stresses of the cycle is called the range of stress. One-half of the stress range, or the algebraic half-difference between the maximum and minimum stresses of the cycle, is called the stress amplitude and denoted by σ_a, or τ_a for shearing stresses

$$\sigma_a = \frac{\sigma_{max}-\sigma_{min}}{2}, \qquad \tau_a = \frac{\tau_{max}-\tau_{min}}{2}. \tag{14.12}$$

The stress ratio (cycle ratio) is defined as the ratio of the minimum to the maximum stress taken with the proper signs

$$r = \frac{\sigma_{min}}{\sigma_{max}}, \qquad r = \frac{\tau_{min}}{\tau_{max}}. \tag{14.13}$$

If the maximum and minimum stresses are equal in magnitude but opposite in sign, i. e., if $r = -1$, the stress cycle is termed completely reversed (Fig. 199). For a completely reversed stress cycle the mean stress is zero ($\sigma_m = 0$, $\tau_m = 0$). If the minimum stress is zero, i. e., $r = 0$, the stress cycle is referred to as pulsating (Fig. 200). If the maximum and minimum stresses are unequal in magnitude, the stress cycle is said to be a fluctuating stress cycle. Figure 198 shows a fluctuating stress cycle with a positive mean stress.

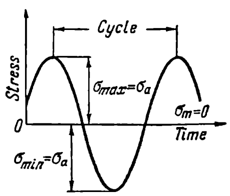


Fig. 199

It is customary to consider the tensile stress positive and the compressive stress negative. The sign of stresses in the case of shearing stresses is

assumed arbitrarily: the stress in one direction is considered positive, and the stress in the opposite direction negative. As can easily be seen, any cycle of varying stresses can be obtained by superimposing a completely reversed (or alternating) stress on a constant

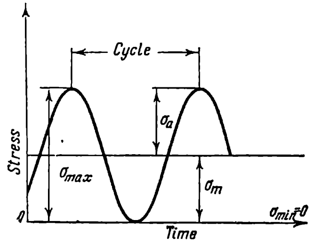


Fig. 200

(or steady) mean stress σ_m. The maximum and minimum stresses are expressed in this case by the formulas

$$\begin{aligned} \sigma_{\max} &= \sigma_m + \sigma_a, \\ \tau_{\max} &= \tau_m + \tau_a, \\ \sigma_{\min} &= \sigma_m - \sigma_a, \\ \tau_{\min} &= \tau_m - \tau_a. \end{aligned} \tag{14.14}$$

In the design of machine parts and structures subjected to varying stresses the basic strength characteristic of the material is the fatigue limit or the endurance limit. *The fatigue (endurance) limit for a given stress ratio r is defined as the maximum value of stress which the material is capable of sustaining for an unlimited number of cycles.* If, when speaking of the fatigue limit, the stress ratio r is not specified, a completely reversed stress cycle is implied.

The purpose of fatigue testing of materials is to determine fatigue (endurance) limits and to evaluate the effect of various factors on them.

100. Fatigue Testing of Materials

The fatigue (endurance) limits of a material corresponding to different values of the mean stress are different. For a completely reversed stress cycle ($\sigma_m = 0$), i.e., when the stress varies between two extreme values equal in magnitude and opposite in sign, the fatigue limit of the material is of minimum value. The determina-

tion of the fatigue limit for the completely reversed stress cycle, which is the most dangerous one, is of the utmost practical interest. Hence the fatigue (endurance) limit is commonly determined for this most dangerous stress cycle.

Fatigue tests of materials are usually carried out in testing machines providing repeated loads of 2,000 to 3,000 cycles per minute. Sometimes use is made of machines with considerably higher frequency of loading, of the order of tens of thousands of cycles per minute.

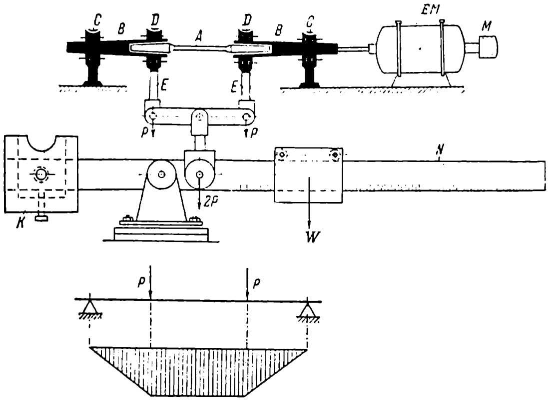


Fig. 201

A schematic diagram of one of the commonly used machines for fatigue testing in bending is shown in Fig. 201.

A test piece A together with tapered grips B forms a beam simply supported in fixed bearings C.

Two free bearings D are connected to a lever system by means of tie rods E. A weight W can be moved along lever N and thereby change the load acting on the test piece.

Weight K balances the system when weight W is in the zero position. As is easily seen from the moment diagram, the specimen is subjected to pure bending. The bending moment is constant throughout the length of the specimen. The test piece with the grips is rotated by the electric motor EM. Due to the rotation, the stress induced in the specimen varies though the load remains constant. The number of revolutions is registered by the counter M. If the

specimen breaks, the lever N comes down and the electric motor and the revolution counter are cut off automatically.

To determine the endurance limit for the completely reversed stress cycle a series of tests have to be performed on 6 to 8 identical specimens made from a given material and thoroughly finished. The first specimen is mounted in the machine and loaded by completely reversed (alternating) stresses. The stress amplitude for the first specimen is generally taken equal to 0.5 to 0.6 of the ultimate strength of the given material. After a certain number of cycles registered by the counter, the first specimen breaks and the machine is shut down automatically. Then a second specimen is taken and loaded with a stress amplitude lower than for the first specimen. After the failure of the second specimen, which is tested at a lower stress and therefore breaks after a larger number of cycles, a third specimen is mounted and loaded with a stress amplitude lower than for the second specimen, and so on. The test is completed when, as a result of a gradual decrease of the stress amplitude and an increase of the number of cycles, we find such a stress amplitude at which the succeeding specimen does not break after a large number of cycles (about 10 million cycles).

Experiments show that if a steel specimen has not broken after 10 million cycles, it can withstand an arbitrarily large number of cycles without fracture. This is not true for non-ferrous metals. A specimen made of non-ferrous metal and subjected to 10 million cycles may fail at the imposed stress amplitude if a larger number of cycles is applied. Therefore, in determining the fatigue limit of non-ferrous metals the specimen is considered to withstand safely a given stress amplitude if it has been subjected to 20×10^7 to 50×10^7 cycles.

After carrying out the foregoing tests the results are represented graphically as a curve to determine the numerical value of the endurance limit.

Figure 202 shows such a curve. The ordinates of this curve are the stress amplitudes imposed on the specimens and the abscissas are the numbers of cycles corresponding to these amplitudes. These curves are often called Wohler's curves after the name of one of the founders of the science of fatigue of materials. The endurance limit is determined as the constant ordinate of the portion of the curve where it becomes parallel to the axis of abscissas.

Stresses above the endurance limit, which the material can withstand for only a limited number of cycles, are called endurance limits for a given number of cycles.

Fatigue limits are determined for different types of deformation: tension-compression, alternating bending and alternating torsion.

Because of their greater simplicity bending stress machines are the most commonly used in practice and alternating bending tests

are more frequently performed than tension-compression or torsion tests.

On the basis of a great body of data for steels the following approximate relations have been established between the fatigue

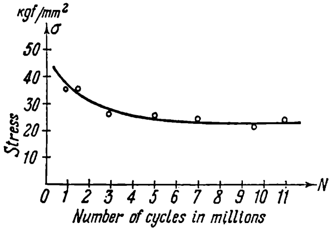


Fig. 202

limit in bending and the fatigue limits under other types of deformation

$$\sigma_{-1t} = 0.7\sigma_{-1}, \qquad (14.15)$$

$$\tau_{-1} = 0.58\sigma_{-1}, \qquad (14.16)$$

where σ_{-1t}, τ_{-1} and σ_{-1} are the fatigue limits for a completely reversed stress cycle in tension-compression, torsion and bending, respectively.

The fatigue limits of steels for completely reversed stress cycles can be estimated from the known ultimate strength of the material using the following empirical relations

$$\sigma_{-1t} = 0.28\sigma_u, \qquad (14.17)$$

$$\sigma_{-1} = 0.4\sigma_u, \qquad (14.18)$$

$$\tau_{-1} = 0.22\sigma_u, \qquad (14.19)$$

101. Endurance Limit for Fluctuating Stress Cycle

As the mean stress increases, the fatigue limit of the material rises and the stress amplitude that the material can withstand without failure decreases. We shall illustrate this by an example. Consider two kinds of stress cycle (completely reversed and pulsating) for a steel containing 0.45 per cent of carbon. In the case of a completely reversed stress cycle of alternating tension and compression when $\sigma_m = 0$ the fatigue limit of this steel is $\sigma_{-1t} = 2{,}000$ kgf/cm².

In the case of a pulsating stress cycle the fatigue limit is $\sigma_0 = 3{,}600$ kgf/cm², i. e., for a mean stress of $\sigma_m = 1{,}800$ kgf/cm² the

stress amplitude that can be imposed is $\sigma_a = 1{,}800$ kgf/cm². Consequently, as the mean stress increases by 1,800 kgf/cm², the fatigue limit increases from 2,000 to 3,600 kgf/cm², i. e., by 1,600 kgf/cm², but the stress amplitude decreases from 2,000 to 1,800 kgf/cm², i. e., by 200 kgf/cm². Thus, an increase in the mean stress of 1,800 kgf/cm² is achieved in this case at the cost of reducing the stress amplitude by 200 kgf/cm².

No general relation exists which would allow one to determine how the fatigue limit increases and the limiting stress amplitude decreases with increasing mean stress.

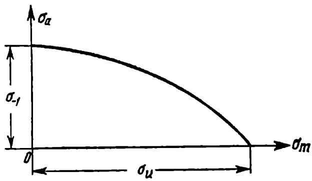


Fig. 203

Therefore, to give a full picture of the behaviour of a material subjected to varying stresses, its fatigue limits are determined at various mean stresses. The test results are represented as diagrams plotted by different methods.

Figure 203 shows a diagram plotted by one of the most frequently used methods. The abscissa is the mean stress of the cycle, σ_m, and the ordinate is the limiting amplitude of the cycle, σ_a. Here the curve represents the relation between limiting stress amplitudes and mean stresses. Any stress cycle may be characterized by the co-ordinates (σ_m, σ_a) of a point on the curve. The sum of the coordinates, $\sigma_m + \sigma_a$, of any point of the limiting strength curve yields the value of the endurance limit for a given mean stress. Stress cycles represented by points within the region bounded by the axis of abscissas, the axis of ordinates and the limiting strength curve are safe stress cycles. Given such a diagram for a specific material, it is easy to determine the stress amplitude that the material can withstand without failure under a particular mean stress.

102. Effect of Overall Dimensions of Parts on Endurance Limit

Handbooks usually give data on fatigue limits of materials as determined on laboratory specimens of small diameters (5 to 12 mm). Tests conducted on specimens of larger diameters (40 to 50 mm)

and large-sized parts show that the fatigue limit is influenced by the overall dimensions of specimens. The fatigue limit decreases with increasing dimensions. The reduction is most pronounced for specimens having diameters under 100 mm.

Table 11. Factors ε

Specimen diameter, mm	10	20	30	40	50	60	80	100	150	200
ε	1	0.93	0.87	0.82	0.78	0.75	0.70	0.65	0.58	0.55

Further increase of the dimensions of specimens has little effect on the fatigue limit.

Table 11 presents approximate data on the reduction of the fatigue limit of steels with increasing overall dimensions of specimens. In this table the fatigue limit for a specimen of diameter 10 mm is taken as unity, the factor ε is the ratio of the fatigue limit of a specimen of given diameter to the fatigue limit of the specimen of 10-mm diameter.

The reduction in fatigue limit with increasing dimensions of specimens or parts is due to a number of causes. Without discussing these causes we only note that this fact must be taken into consideration in the design of structural elements.

As numerous experiments and observations show localized stresses caused by an abrupt change in section (notches, grooves, fillets, keyways, drilled holes, etc.) or surface damage (scratches, coarse finishing marks) considerably reduce the fatigue limit of steel. An inadequate consideration of this fact is often the cause of failure of various machine parts with abrupt changes in section which create stress concentration.

The higher the ultimate strength of steel, the greater is the effect of localized stresses, i. e., the more drastic is the reduction in the fatigue limit. Thus, the fatigue limit of a steel specimen with an ultimate strength of 5,000 kgf/cm^2 whose surface is roughly tooled is 20 per cent lower than that for a specimen with polished surface, while in the case of a steel specimen having an ultimate strength of 10,000 kgf/cm^2 the reduction is as high as 40 per cent.

For small laboratory specimens with stress concentration, particularly those made of mild steel, the reduction in the fatigue limit is found to be smaller than would be predicted from the value of the stress concentration factor. This will be illustrated by an example.

Suppose we have two laboratory specimens of ordinary dimensions (of diameter 8 to 12 mm) and of the same material; the first specimen is smooth, the second specimen has a groove to which corresponds a stress concentration factor $\alpha = 2$. If the fatigue limit for the first specimen is $\sigma_{-1} = 2{,}000$ kgf/cm², then for the second specimen it is not equal to half this value, i. e., to 1,000 kgf/cm², but is somewhat higher, say, 1,200 kgf/cm².

If the dimensions of the second specimen are increased, its fatigue limit will decrease. The ratio of the fatigue limit of a smooth laboratory specimen for the completely reversed stress cycle to that of a large specimen (or part) with stress concentration is called the effective stress concentration factor and denoted by α_{eff}. The value of the effective stress concentration factor depends not only on the value of the stress concentration factor α but also on the material and the overall dimensions of a specimen or part. The effective stress concentration factor increases with increasing strength of steel and increasing dimensions of parts. For large-sized parts made of high-strength steel (alloy steel or heat-treated carbon steel) the effective stress concentration factor is close to the theoretical value, i.e., if the fatigue limit of a smooth small-diameter specimen of high-strength steel for the completely reversed stress cycle is $\sigma_{-1} = 5{,}100$ kgf/cm², a large-sized specimen of the same steel with a small lateral drilled hole and a stress concentration factor $\alpha = 3$ will have a fatigue limit close to 1,700 kgf/cm². Thus, in choosing the material for parts subjected to fluctuating loads it should be remembered that the higher the strength of steel, the more sensitive it is to stress concentration. Therefore, high-strength steels require more careful surface finish.

Note that cast iron is less sensitive to stress concentration. This advantageous property of cast iron is utilized in such parts as cast crankshafts and the like.

On the basis of fatigue testing of parts and large specimens with different stress concentrations the values of effective stress concentration factors have been determined for cases that occur most frequently in practice. In Sec. 105 are given some values of effective stress concentration factors.

103. Strength Design for Completely Reversed Stresses

In the case of varying stresses the critical limiting stress is taken as the endurance limit of the material determined on laboratory specimens. The endurance limit of the material depends on the stress ratio. Therefore, we begin with the particular case when the variable stresses are completely reversed. In this simplest case the allowable stress (the allowable stress amplitude) for a part

without stress concentration is defined by the formula

$$[\sigma_a]=\frac{\sigma_{-1}\varepsilon}{k}, \tag{14.20}$$

where σ_{-1} is the endurance limit of the material for the completely reversed stress cycle determined on ordinary laboratory specimens; ε is the size effect factor; the numerical value of this factor is taken from Table 11; k is the factor of safety.

When stress concentration is present, the allowable stress in the case of complete reversal of stresses is defined by the formula

$$[\sigma_a]=\frac{\sigma_{-1}}{\alpha_{eff}k}, \tag{14.21}$$

where α_{eff} is the effective stress concentration factor.

Below are given several examples of designing for completely reversed stresses.

Example 89. Determine the ratio of the diameters of two smooth bars if the first bar is acted on by a static tensile force P, and the second, by a force of equal magnitude which alternately compresses and stretches the bar. The yield strength of the material is $\sigma_y=3{,}600$ kgf/cm², the fatigue limit $\sigma_{-1}=2{,}000$ kgf/cm². The factor of safety k must be the same for the two bars.

Solution. The diameter of the first bar is

$$d_1=\sqrt{\frac{4Pk}{\pi\sigma_y}}.$$

The diameter of the second bar is

$$d_2=\sqrt{\frac{4Pk}{\pi\sigma_{-1}}}.$$

The required ratio of the diameters is

$$\frac{d_1}{d_2}=\sqrt{\frac{\sigma_{-1}}{\sigma_y}}=\sqrt{\frac{2{,}000}{3{,}600}}\cong 0.75.$$

Example 90. Determine the allowable value of a force acting on a connecting rod if it changes from $+P$ in one direction to $-P$ in the opposite direction. The cross section of the connecting rod is circular, 40 mm in diameter, the fatigue limit of the material is $\sigma_{-1}=2{,}000$ kgf/cm². Take a factor of safety $k=2$.

Solution. If the fatigue limit $\sigma_{-1}=2{,}000$ kgf/cm², the fatigue limit of a 40-mm diameter part is, according to Table 11,

$$\sigma_{-1}^{p}=\varepsilon\sigma_{-1}=0.82\times 2{,}000=1{,}640 \text{ kgf/cm}^2.$$

The allowable stress amplitude is

$$[\sigma_a]=\frac{1{,}640}{2}=820 \text{ kgf/cm}^2.$$

The allowable force amplitude is

$$P_a = [\sigma_a]\, A = 820\,\frac{3.14}{4}\,4^2 = 10{,}170 \quad \text{kgf}.$$

Example 91. Determine the allowable angle of twist of a circular steel bar in either direction if the torsional loading is repeatedly applied for an infinite number of times. The diameter of the bar is $d = 12$ mm, its length $l = 1.5$ m, the modulus of elasticity in shear $G = 0.8 \times 10^6$ kgf/cm^2, the fatigue limit of the material $\tau_{-1} = 1{,}500$ kgf/cm^2, the factor of safety $k = 1.8$.

Solution. The allowable stress amplitude is

$$[\tau_a] = \frac{\tau_{-1}}{k}.$$

Since the diameter of the bar is small, the effect of overall dimensions on τ_{-1} is disregarded.

The allowable amplitude of the torque is

$$(M_t)_a = [\tau_a]\, Z_p = \frac{\tau_{-1}}{k}\, Z_p.$$

Substituting this value of the torque in formula (6.6), we find the allowable amplitude of the angle of twist

$$\varphi_a^\circ = \frac{180^\circ}{\pi}\,\frac{\tau_{-1} Z_p l}{kGI_p} = \frac{180^\circ}{3.14}\,\frac{1{,}500 \times 3.14 \times 1.2^3 \times 150 \times 32}{1.8 \times 16 \times 0.8 \times 10^6 \times 3.14 \times 1.2^4} \cong 15^\circ.$$

Example 92. Determine the wire diameter d of a shock-absorber helical spring acted on by a force $P = 0.4$ ton which alternately stretches and compresses the spring, if the mean radius of the spring is $R = 80$ mm, the fatigue limit of the material in torsion $\tau_{-1} = 3{,}600$ kgf/cm^2 and the factor of safety with respect to failure by fatigue is to be $k = 2$.

Solution. The spring is subjected in this case to completely reversed stresses. Find the allowable stress amplitude

$$[\tau_a] = \frac{\tau_{-1}}{k} = \frac{3{,}600}{2} = 1{,}800 \quad \text{kgf/cm}^2.$$

Since $\tau_{max} = \tau_a$ for a completely reversed stress cycle, we substitute the amplitude τ_a for τ_{max} in formula (6.21) to determine d

$$[\tau_a] = \frac{16PR}{\pi d^3},$$

whence

$$d = \sqrt[3]{\frac{16PR}{\pi[\tau_a]}} = \sqrt[3]{\frac{16 \times 400 \times 8}{3.14 \times 1{,}800}} \cong 2.09 \quad \text{cm}.$$

Take $d = 21$ mm.

104. Determination of Factor of Safety in the Case of Fluctuating Stresses

Consider now fluctuating stresses occurring in a part. In this case the problem of determining the factor of safety or allowable stresses is complicated by the fact that it is necessary to take not a single quantity defining the limiting state, as in the case of constant or completely reversed stresses, but two quantities. The limiting stress under constant stresses is taken as the ultimate strength or the yield strength, and as the fatigue limit for the completely reversed stress cycle (σ_{-1}) under completely reversed stresses; the limiting state under fluctuating stresses is characterized by two quantities, namely the mean stress and the corresponding limiting amplitude. Therefore, the determination of the factor of safety or allowable stresses in the case of fluctuating stresses imposed on a part is somewhat arbitrary. It is customary to take the limiting or failure cycle as a cycle with a stress ratio (r) equal to that for the part. Such cycles, i.e., cycles with equal stress ratios are called similar cycles.

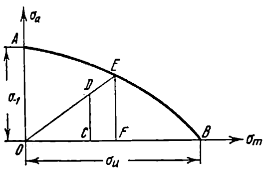


Fig. 204

By way of illustration we recall the complete fatigue strength diagram (Fig. 204) which gives the relation between limiting stress amplitudes and mean stresses. Any stress cycle is characterized in this diagram by the co-ordinates of a point (σ_m, σ_a).

Stress cycles represented by points within the region bounded by straight lines OA, OB and curve AB are safe stress cycles. Points on the curve AB represent limiting cycles. Suppose that the stress cycles in a part are represented by point D, i.e., the mean stress in the part is $\sigma_m=\overline{OC}$ and the stress amplitude is $\sigma_a=\overline{DC}$. The limiting cycle is then represented by the point of intersection E of ray OD and curve AB. Point E represents a cycle with the same stress ratio as for the cycle represented by point D. Indeed, the stress ratio is defined as $r=\sigma_{min}/\sigma_{max}$. For the cycle represented by point D, the minimum and maximum stresses are, respectively,

$$\sigma_{min}=\overline{OC}-\overline{CD}, \qquad \sigma_{max}=\overline{OC}+\overline{CD}.$$

Consequently,

$$r_D=\frac{\overline{OC}-\overline{CD}}{\overline{OC}+\overline{CD}}.$$

For the cycle represented by point E, the same quantities are, respectively,

$$\sigma_{\min}=\overline{OF}-\overline{FE}, \qquad \sigma_{\max}=\overline{OF}+\overline{FE},$$
$$r_E=\frac{\overline{OF}-\overline{FE}}{\overline{OF}+\overline{FE}}.$$

From the similar triangles ODC and OEF it follows that

$$\frac{\overline{OC}}{\overline{CD}}=\frac{\overline{OF}}{\overline{FE}} \quad \text{or} \quad \frac{\overline{OC}-\overline{CD}}{\overline{OC}+\overline{CD}}=\frac{\overline{OF}-\overline{FE}}{\overline{OF}+\overline{FE}}.$$

Consequently, $r_D=r_E$, i.e., the limiting cycle represented by point E has the same stress ratio as the cycle represented by point D, i. e., these cycles are similar. In general, points on any ray drawn from the origin represent similar cycles.

In designing parts subjected to fluctuating stresses it is customary first to assign the dimensions of parts. Next the stresses and the resulting factor of safety are determined from these dimensions and loads. If the factor of safety is found to be inadequate, the dimensions of parts are increased and the factor of safety is determined again. Thus, an analysis under fluctuating stresses is usually made as a checkup. This is due to the fact that the determination of the dimensions of a part from the allowable stresses (mean stress and stress amplitude) requires a knowledge of the values of the allowable stresses which themselves depend on the stress ratio r.

Consequently, in this case it is necessary to assign a stress ratio, which is not always easy to do. The factor of safety is, in general, the ratio of the stress at failure to the stress in a part. Given a complete fatigue strength diagram, the factor of safety for the fluctuating stress cycle is easily determined from the ratio of the stresses of the limiting cycle to the stresses in a part. If the limiting cycle is taken as a similar cycle, it is immaterial what stresses of the two cycles are compared when determining the factor of safety. The factor of safety will be the same whether we take the ratio of the maximum stress of the limiting cycle to the maximum stress in the part, the ratio of the amplitudes of the two cycles, or the ratio of their mean stresses, i. e., the factor of safety k will be

$$k=\frac{\overline{OF}+\overline{FE}}{\overline{OC}+\overline{CD}}=\frac{\overline{FE}}{\overline{CD}}=\frac{\overline{OF}}{\overline{OC}}.$$

In this way the factor of safety is determined if the fatigue strength diagram of a part is available.

105. Construction of Approximate Fatigue Strength Diagram and Determination of Factor of Safety from It

In practice, it is only in rare cases that we have the fatigue strength diagram of a part at our disposal to determine the factor of safety. In many cases we even lack a complete fatigue strength diagram of the material which is obtained from tests on laboratory specimens with different stress ratios. This is due to the long duration of the tests and a relatively small number of machines available in which such tests can be performed. Therefore, approximate fatigue strength diagrams are often used in design practice.

To construct approximate diagrams, we need to make use of one or another presumed relation between the limiting amplitude and the mean stress of the cycle. In these relations, the limiting amplitude is expressed in terms of the fatigue limit σ_{-1} (for the completely reversed stress cycle) and σ_u or σ_y.

Comparison of experimental diagrams with approximate ones constructed on the basis of the proposed relations between the limiting amplitude and the mean stress of the cycle in which the limiting amplitude is expressed in terms of a single fatigue characteristic (σ_{-1}) shows that none of these relations can be considered satisfactory for a wide range of steels.

In order to construct a more or less accurate approximate diagram, it is obviously insufficient to have one fatigue characteristic, σ_{-1}.

To construct approximate diagrams and to find factors of safety from them, the author proposed a relation for determining the limiting amplitude as a function of the mean stress, which includes σ_{-1} and a second fatigue characteristic of the material, namely the fatigue limit for the pulsating stress cycle σ_0.

Diagrams constructed on the basis of this relation are found to be fairly close to experimental diagrams. A disadvantage of this method is that the construction requires a knowledge of the second experimental fatigue characteristic, σ_0. Nevertheless, the proposed method of constructing an approximate diagram and the method of determining factors of safety from it have received wide acceptance in this country and therefore we present them here.

Suppose that we know the following strength characteristics for a particular steel: the fatigue limit for the completely reversed stress cycle σ_{-1}, the fatigue limit for the pulsating stress cycle σ_0 and the yield strength σ_y. Consider separately the cases when $\sigma_0 < \sigma_y$ and when $\sigma_0 > \sigma_y$.

1. *Construction of Approximate Diagram When* $\sigma_0 < \sigma_y$ (Fig. 205). On the axis of ordinates we mark a point A with co-ordinates $\sigma_m = 0$, $\sigma_a = \sigma_{-1}$, which represents a completely reversed stress cycle. Mark a point B representing a pulsating stress cycle. For a pulsating stress cycle, $\sigma_{min} = 0$, $\sigma_{max} = \sigma_0$; consequently, the co-ordinates

of point B are

$$\sigma_m = \frac{\sigma_0}{2}, \quad \sigma_a = \frac{\sigma_0}{2}.$$

Mark a point C with co-ordinates $\sigma_m = \sigma_y$, $\sigma_a = 0$, which represents a constant stress equal to the yield point stress. Through points A and B we draw a straight line until it intersects, at point D,

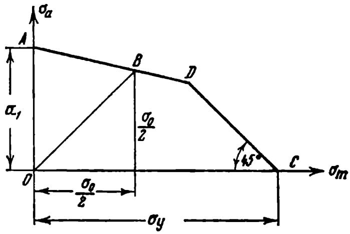


Fig. 205

a straight line drawn from point C at an angle of 45° to the axis of abscissas. Points lying on line CD represent cycles with maximum stresses equal to the yield stress. This follows from the fact that the sum of the co-ordinates of these points is equal to σ_y. Thus the

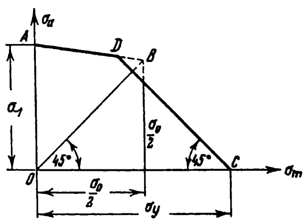


Fig. 206

ordinates of the broken line ADC represent the limiting stress amplitudes. Since the maximum stresses must not be higher than the yield stress, the fatigue strength diagram is limited by the maximum stress equal to the yield stress.

2. *Construction of Approximate Diagram When* $\sigma_0 > \sigma_y$ (Fig 206). As in the first case, we mark points A, B and C. Through points

A and B we draw a straight line until it intersects, at point D, a straight line drawn from point C at an angle of 45° to the axis of abscissas. The ordinates of the broken line ADC represent the limiting stress amplitudes. In this case point B falls outside the limits of the diagram.

Find the relation between the limiting amplitude and the mean stress of the cycle from the approximate diagram.

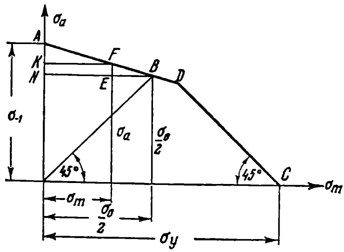


Fig. 207

Take any limiting stress cycle represented, say, by point F (Fig. 207). From the similar triangles AKF and ANB we find

$$\frac{\overline{AK}}{\overline{AN}}=\frac{\overline{KF}}{\overline{NB}} \quad \text{or} \quad \frac{\sigma_{-1}-\sigma_a}{\sigma_{-1}-\frac{\sigma_0}{2}}=\frac{\sigma_m}{\frac{\sigma_0}{2}},$$

whence

$$\sigma_a=\sigma_{-1}-\frac{2\sigma_{-1}-\sigma_0}{\sigma_0}\sigma_m. \qquad (14.22)$$

This formula allows a sufficiently accurate determination of the limiting stress amplitude for a given mean stress when σ_{-1} and σ_0 are known.

For brevity, we denote the factor in front of σ_m in formula (14.22) by a single letter

$$\varphi_\sigma=\frac{2\sigma_{-1}-\sigma_0}{\sigma_0}. \qquad (14.23)$$

Formula (14.22) is then rewritten as

$$\sigma_a=\sigma_{-1}-\varphi_\sigma\sigma_m. \qquad (14.24)$$

Rewrite formula (14.24) as follows:

$$\sigma_{-1}=\sigma_a+\varphi_\sigma\sigma_m. \qquad (14.25)$$

From this it is seen that the factor φ_a may be regarded as a factor reducing a fluctuating stress cycle to an equally dangerous completely reversed stress cycle.

The factor of safety is determined from the formula

$$k=\frac{\sigma_{-1}}{\sigma_a'+\varphi_a\sigma_m'}, \tag{14.26}$$

where σ_a' and σ_m' are the working stresses for a part.

Note that if the factor of safety were determined from this formula for all stress cycles, then for some cycles it would be larger than if determined as the ratio of the yield stress to the maximum stress, i. e., from the formula

$$k=\frac{\sigma_y}{\sigma_a'+\sigma_m'}. \tag{14.27}$$

This means that for some operating conditions the limiting criterion of failure is the fatigue limit, while for other operating conditions it is the yield strength.

Therefore, to find the minimum factor of safety, we have to determine it from formulas (14.26) and (14.27).

It may be stated a priori which of the formulas gives the minimum factor of safety but this question will not be discussed here since the solution of this problem is as time-consuming as a direct comparison of the values of safety factors determined from the two formulas, (14.26) and (14.27).

The factor of safety with respect to failure by fatigue for parts without stress concentration but of larger dimensions than laboratory specimens used to determine σ_{-1} and σ_0 is found, with due allowance for the size effect, from the formula

$$k=\frac{\sigma_{-1}}{\frac{\sigma_a'}{\varepsilon}+\varphi_a\sigma_m'}, \tag{14.28}$$

where ε is a factor which accounts for the reduction in the fatigue limit with increase in dimensions; this factor is taken from Table 11, p. 322, i.e., the size effect factor applies only to the variable part of the stress. The factor of safety with respect to failure by fatigue for parts with stress concentration is determined from σ_{-1} and σ_0 found for smooth laboratory specimens by the formula

$$k=\frac{\sigma_{-1}}{\sigma_a'\alpha_{eff}+\varphi_a\sigma_m'}, \tag{14.29}$$

where α_{eff} is the effective stress concentration factor. It will be recalled that the effective stress concentration factor is defined as the ratio of the fatigue limit of the material for the completely

reversed stress cycle found by testing laboratory specimens to the fatigue limit of the part for the completely reversed stress cycle.

Note that the absolute value of the mean stress, $|\sigma_m|$, is substituted in all the formulas defining the factor of safety.

Assigning a certain factor of safety, it is easy to obtain the allowable stresses from formula (14.29) for various values of the ratio σ_a'/σ_m'.

The allowable nominal stresses are

$$[\sigma_m]=\frac{\sigma_{-1}}{k}\,\frac{1}{\dfrac{\sigma_a'\alpha_{\text{eff}}}{\sigma_m'}+\varphi_\sigma}, \tag{14.30}$$

$$[\sigma_a]=\frac{\sigma_{-1}}{k}\,\frac{1}{\alpha_{\text{eff}}+\varphi_\sigma\dfrac{\sigma_m'}{\sigma_a'}}. \tag{14.31}$$

All the foregoing formulas derived for normal stresses are valid in the case of shearing stresses. Formulas for shearing stresses may be written by analogy with those given above; for this purpose σ_{-1}, σ_0, σ_m', σ_a', φ_σ should be replaced by τ_{-1}, τ_0, τ_m', τ_a', φ_τ,

Table 12

Steel	σ_u, $\frac{\text{kgf}}{\text{mm}^2}$	Bending			Tension, compression			Torsion		
		σ_y, $\frac{\text{kgf}}{\text{mm}^2}$	σ_{-1}, $\frac{\text{kgf}}{\text{mm}^2}$	φ_σ	σ_y, $\frac{\text{kgf}}{\text{mm}^2}$	σ_{-1}, $\frac{\text{kgf}}{\text{mm}^2}$	φ_σ	τ_y, $\frac{\text{kgf}}{\text{mm}^2}$	τ_{-1}, $\frac{\text{kgf}}{\text{mm}^2}$	φ_τ
Carbon steel	37	26	17	0.13	22	12	0.09	14	10	0
	45	30	19	0.06	25	14	0.08	17	12	0
	55	37	24	0.09	31	18	0.06	19	14	0.08
	65	43	28	0.17	36	20	0.11	22	15	0
	75	50	33	0.25	42	23	0.12	26	19	0
Chrome-nickel steel	83	69	36	0.31	62	28	0.30	38	20	0.08
	98	81	41	0.23	73	30	0.13	42	24	0.14
	115	101	51	0.22	92	35	0.17	54	29	0.16
Chrome-nickel-tungsten steel	120	109	53	0.22	100	37	0.14	61	30	0.05

respectively, where

$$\varphi_\tau = \frac{2\tau_{-1} - \tau_0}{\tau_0} \, . \tag{14.32}$$

Values of σ_{-1}, τ_{-1}, φ_σ and φ_τ for several steels and different types of deformation are given in Table 12.

In tables that follow are given some values of effective stress concentration factors.

1. *Stepped shaft.* D = larger diameter, d = smaller diameter, r = fillet radius, $D:d = 1.2$, $d = 30$ to 50 mm.

The nominal stresses are

$$\sigma = \frac{M_b}{0.1d^3}, \qquad \tau = \frac{M_t}{0.2d^3} \, .$$

$\frac{r}{d}$	Bending		Torsion	
	$\sigma_u \leqslant 50$, kgf/mm²	$\sigma_u \leqslant 120$, kgf/mm²	$\sigma_u \leqslant 50$, kgf/mm²	$\sigma_u \leqslant 120$, kgf/mm²
0.05	1.8	1.9	1.5	1.6
0.10	1.4	1.5	1.3	1.4
0.15	1.3	1.4	1.2	1.3
0.20	1.2	1.3	1.1	1.2

2. *Shaft with a lateral hole.* d = shaft diameter, a = hole diameter, $d = 40$ to 50 mm.

The nominal stresses are

$$\sigma = \frac{M_b}{Z_{\text{net}}} \cong \frac{M_b}{0.1d^3\left(1 - 1.5\frac{a}{d}\right)},$$

$$\tau = \frac{M_t}{Z_{p\text{net}}} \cong \frac{M_t}{0.2d^3\left(1 - \frac{a}{d}\right)} \, .$$

$\frac{a}{d}$	Bending		Torsion	
	$\sigma_u \leqslant 50$, kgf/mm²	$\sigma_u \leqslant 120$, kgf/mm²	$\sigma_u \leqslant 50$, kgf/mm²	$\sigma_u \leqslant 120$, kgf/mm²
0.05	2.2	2.5	1.8	2.0
0.10	1.9	2.3	1.8	2.0
0.20	1.7	2.0	1.8	2.0

3. *Flat strip with a lateral hole.* B = strip width, a = hole diameter, $a/B = 0.05$ to 0.30.

The nominal stresses are

$$\sigma = \frac{P}{A_{\text{net}}}, \qquad \sigma = \frac{M_b}{Z_{\text{net}}}.$$

σ_u, kgf/mm²	Tension	Bending
40	1.4	1.3
80	1.8	1.6
120	2.0	1.8

4. *Shaft with a keyway.* d = shaft diameter, b = keyway width, t = keyway depth, $d = 100$ mm.

The nominal stresses are

$$\sigma \cong \frac{M_b}{0.1d^3 - \frac{bt(d-t)^2}{2d}}, \qquad \tau \cong \frac{M_t}{0.2d^3 - \frac{bt(d-t)^2}{2d}}.$$

σ_u, kgf/mm²	Bending	Torsion
40	1.6	1.3
80	2.2	1.9
100	2.5	2.2

5. *Shaft with a circular groove.* d = shaft diameter, r = groove radius.

The nominal stresses are

$$\sigma = \frac{M_b}{0.1d^3}, \qquad \tau = \frac{M_t}{0.2d^3}$$

$\frac{r}{d}$	Bending		Torsion	
	$\sigma_u \leqslant 50$, kgf/mm²	$\sigma_u \leqslant 120$, kgf/mm²	$\sigma_u \leqslant 50$, kgf/mm²	$\sigma_u \leqslant 120$, kgf/mm²
0.05	1.8	2.2	1.7	2.1
0.10	1.7	1.8	1.5	1.7
0.20	1.4	1.5	1.4	1.5
0.30	1.3	1.3	1.3	1.3

6. *Shaft with a press-fitted hub transmitting a transverse force to the shaft.* The fit pressure on the hub is p kgf/mm². The shaft diameter is $d=50$ mm.

σ_u, kgf/mm²	$p=1$ kgf/mm²	$p \geqslant 3$ kgf/mm²
40	1.3	1.6
80	2.1	2.6
100	2.6	3.2

7. *Threaded joint of bolt-nut type.*

Type of thread	Carbon steel	Alloy steel
Metric thread	3-4	4-5
Whitworth thread	3-4.5	4-5.5

106. Determination of Factor of Safety in the Case of Combined Varying Stresses

The combined stresses that occur most frequently in practice are produced by a combination of torsion and bending or tension (compression). Another common case when tensile (compressive) stress is combined with bending stress is reduced to the case of fluctuating stresses considered above.

In the case of combined stresses the factor of safety is determined from the formula

$$\left(\frac{\sigma'_a+\varphi_\sigma\sigma'_m}{\sigma_{-1}}\right)^2+\left(\frac{\tau'_a+\varphi_\tau\tau'_m}{\tau_{-1}}\right)^2=\frac{1}{k^2}, \tag{14.33}$$

where σ'_a, σ'_m, τ'_a and τ'_m are the working stresses for a part.

Denoting the factor of safety based on normal stresses by k_σ and the factor of safety based on shearing stresses by k_τ, formula (14.33) can be rewritten as

$$\left(\frac{1}{k_\sigma}\right)^2+\left(\frac{1}{k_\tau}\right)^2=\left(\frac{1}{k}\right)^2$$

or

$$k=\frac{k_\sigma k_\tau}{\sqrt{k_\sigma^2+k_\tau^2}}. \tag{14.34}$$

Factors of safety based on normal and shearing stresses are determined from formulas given in the preceding section.

To determine the minimum overall factor of safety, the minimum values of k_σ and k_τ should be substituted in formula (14.34). It will be recalled that for some stress cycles the factor of safety, if determined as the ratio of the yield stress to the maximum stress, may be smaller than the factor of safety with respect to failure by fatigue.

107. Examples of Design for Varying Stresses

Example 93. Determine the factor of safety for the connecting rod of an engine. The rod diameter is $d = 60$ mm. At the moment of firing in the cylinder a compressive force of 52 tons acts along the axis of the connecting rod, and at the beginning of intake a tensile force of 12 tons. The fatigue limit of the material for the completely reversed stress cycle is $\sigma_{-1t} = 2{,}900\ \text{kgf/cm}^2$, the yield strength $\sigma_y = 5{,}000\ \text{kgf/cm}^2$, $\varphi_\sigma = 0.16$.

Solution. Find the minimum and maximum stresses

$$\sigma_{\min} = -\frac{52{,}000}{3.14\times 6^2} = -1{,}840\ \text{kgf/cm}^2,$$

$$\sigma_{\max} = \frac{12{,}000\times 4}{3.14\times 6^2} = 425\ \text{kgf/cm}^2.$$

Find the mean stress and the stress amplitude

$$\sigma'_m = \frac{425+(-1{,}840)}{2} = -708\ \text{kgf/cm}^2,$$

$$\sigma'_a = \frac{425-(-1{,}840)}{2} = 1{,}132\ \text{kgf/cm}^2.$$

Determine the factor of safety using formula (14.28)

$$k = \frac{\sigma_{-1t}}{\dfrac{\sigma'_a}{\varepsilon} + \varphi_\sigma \sigma'_m}. \tag{14.35}$$

From Table 11, for a 60-mm diameter rod we take the factor $\varepsilon = 0.75$.

We substitute the numerical values in formula (14.28) remembering that σ'_m is replaced by its absolute value

$$k = \frac{2{,}900}{\dfrac{1{,}132}{0.75} + 0.16\times 708} = 1.8.$$

Find the factor of safety with respect to failure by yielding from formula (14.27)

$$k = \frac{5{,}000}{1{,}132+708} = 2.7.$$

From comparison of the above safety factors it is apparent that the danger of failure of the connecting rod by fatigue is by far more serious than the danger of failure by yielding.

Example 94. Determine the allowable stresses for a shaft subjected to a variable bending moment. The effective stress concentration factor at the fillet in the shaft is $\alpha_{eff}=1.5$, the yield strength of the material is $\sigma_y=5{,}000$ kgf/cm², the fatigue limit $\sigma_{-1}=3{,}300$ kgf/cm², $\varphi_\sigma=0.25$. The ratio $\sigma'_m/\sigma'_a=1/2$. The factor of safety for the shaft is to be $k=2$.

Solution. Using formulas (14.30) and (14.31), we determine the allowable stresses

$$\sigma_m=\frac{\sigma_{-1}}{k}\,\frac{1}{\dfrac{\sigma'_a}{\sigma'_m}\alpha_{eff}+\varphi_\sigma}=\frac{3{,}300}{2}\,\frac{1}{2\times1.5+0.25}=510\ \text{kgf/cm}^2,$$

$$\sigma_a=\frac{\sigma_{-1}}{k}\,\frac{1}{\alpha_{eff}+\varphi_\sigma\dfrac{\sigma'_m}{\sigma'_a}}=\frac{3{,}300}{2}\,\frac{1}{1.5+0.25\times0.5}=1{,}000\ \text{kgf/cm}^2.$$

Example 95. A hollow shaft of outer diameter $D=80$ mm and inner diameter $d=40$ mm containing a drilled small-diameter lateral hole (oil-hole) is subjected to a variable torque and a variable bending moment. The maximum and minimum values of these moments are the following:

$$M_{t\,max}=24{,}000\ \text{kgf-cm},\quad M_{t\,min}=-6{,}000\ \text{kgf-cm},$$
$$M_{b\,max}=20{,}500\ \text{kgf-cm},\quad M_{b\,min}=-10{,}500\ \text{kgf-cm}.$$

The effective stress concentration factor at the lateral drilled hole is $\alpha_{eff}=3$. The yield strength of the material in bending is $\sigma_y=4{,}300$ kgf/cm², the yield strength in torsion is $\tau_y=2{,}200$ kgf/cm², the fatigue limit in bending is $\sigma_{-1}=2{,}700$ kgf/cm², the fatigue limit in torsion is $\tau_{-1}=1{,}500$ kgf/cm², $\varphi_\sigma=0.17$, $\varphi_\tau=0$.

Solution. Find the section moduli of the shaft

$$Z=\frac{\pi}{32}8^3\left[1-\left(\frac{4}{8}\right)^4\right]=46.1\ \text{cm}^3,$$
$$Z_p=2Z=2\times46.1=92.2\ \text{cm}^3.$$

Find the stresses

$$\tau_{max}=\frac{24{,}000}{92.2}=260\ \text{kgf/cm}^2,$$

$$\tau_{min}=-\frac{6{,}000}{92.2}=-65\ \text{kgf/cm}^2,$$

$$\sigma_{max}=\frac{20{,}500}{46.1}=445\ \text{kgf/cm}^2,$$

$$\sigma_{min}=-\frac{10{,}500}{46.1}=-228\ \text{kgf/cm}^2,$$

$$\tau'_m = \frac{260-65}{2} = 98 \text{ kgf/cm}^2,$$

$$\tau'_a = \frac{260+65}{2} = 163 \text{ kgf/cm}^2,$$

$$\sigma'_m = \frac{445-228}{2} = 108 \text{ kgf/cm}^2,$$

$$\sigma'_a = \frac{445+228}{2} = 336 \text{ kgf/cm}^2.$$

The factor of safety based on shearing stresses is determined by formulas similar to formulas (14.29) and (14.27) for normal stresses

$$k_\tau = \frac{\tau_{-1}}{\tau'_a \alpha_{eff} + \varphi_\tau \tau'_m} = \frac{1{,}500}{163 \times 3} = 3.1.$$

The static factor of safety in torsion is

$$k_\tau = \frac{\tau_y}{\tau'_a + \tau'_m} = \frac{2{,}200}{163+98} = 8.4.$$

The factor of safety based on normal stresses is

$$k_\sigma = \frac{\sigma_{-1}}{\sigma'_a \alpha_{eff} + \varphi_\sigma \sigma'_m} = \frac{2{,}700}{336 \times 3 + 0.17 \times 108} = 2.6.$$

The static factor of safety is

$$k_\sigma = \frac{\sigma_y}{\sigma'_a + \sigma'_m} = \frac{4{,}300}{336+108} = 9.7.$$

Determine the overall factor of safety, substituting the minimum values of k_σ and k_τ in formula (14.34)

$$k = \frac{k_\sigma k_\tau}{\sqrt{k_\sigma^2 + k_\tau^2}} = \frac{2.6 \times 3.1}{\sqrt{2.6^2 + 3.1^2}} = 2.$$

108. Improvement of Fatigue Strength

Most failures of machine parts result from fatigue, therefore measures directed towards increasing the fatigue strength and life of parts are of great practical importance. The basic measures are concerned with design and the manufacturing process. The former are aimed essentially at reducing stress concentrations at locations where fatigue cracks are likely to occur causing structural damage. Consider some examples.

In many cases fatigue cracks occur at places where the section changes abruptly—at fillets in shafts, axles and other parts. Increasing the fillet radius reduces the stress concentration factor and thereby improves the fatigue strength of a part. It is not always possible, however, to increase the fillet radius to a desirable value;

in such exceptional cases a fillet is made with an undercut, as shown in Fig. 208.

Frequently the edges of drilled holes in parts, as in holes for supplying oil to rubbing surfaces, are inadequately rounded off, which considerably reduces the fatigue strength of parts. Figure 209 shows drilled holes with sharp edges and well-rounded edges.

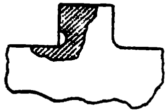


Fig. 208

Hollow shafts containing lateral drilled holes are sometimes provided with an inner flange, as shown in Fig. 210, to strengthen the weakened portion of the shaft.

To reduce the stress concentration caused by a keyway, the re-entrant corner is eliminated by straight milling of the keyway. Figure 211 shows keyways with a re-entrant corner and a gradual transition achieved by straight milling.

The fatigue strength of a shaft subjected to reversed bending is considerably reduced when a hub is mounted on the shaft. This is

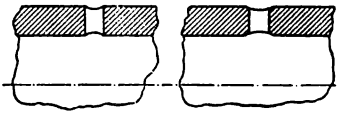


Fig. 209

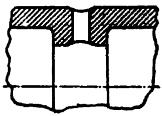


Fig. 210

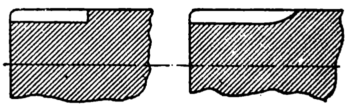


Fig. 211

due to the stress concentration in the shaft at the edges of the hub and fretting corrosion. To improve the fatigue strength, the shaft diameter under the hub is sometimes increased (Fig. 212) or the transition from hub to shaft is made gradual by making an inner chamfer at the edges of the hub (Fig. 213).

A considerable reduction in the fatigue strength of a material results from various scratches on the surface of parts left by rough mechanical treatment. Corrosion greatly reduces the fatigue strength. Figure 214 shows the reduction in fatigue limits of steel specimens due to the above-mentioned factors as a function of the ultimate strength of steel. Fatigue limits of polished specimens (line *a*) are

taken as 100 per cent; curve b = fatigue limits for ground specimens, curve c for turned specimens, curve d for specimens with a sharp notch such as shown in the insert at the lower left of the figure, curve e for specimens in the as-rolled condition, curve f for

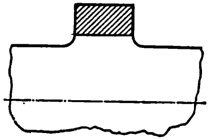


Fig. 212

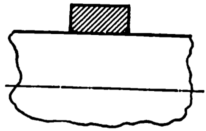


Fig. 213

specimens whose surface is corroded in fresh water, and curve g for specimens whose surface is corroded in sea water. It is seen from Fig. 214 that the higher alloyed the steel and the higher its ultimate tensile strength, the greater is the reduction in its fatigue limit due to surface damage of the specimen by tool marks or corrosion. Therefore, as stated above, the use of expensive steels requires a particularly thorough surface finishing of parts and corrosion protection.

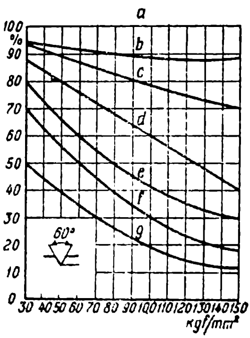


Fig. 214

We now turn to a brief discussion of measures concerned with the manufacturing process which improve the fatigue strength of parts. They reduce essentially to surface hardening of parts.

In the case of frequently encountered bending and torsional loadings the maximum stresses occur in the surface layers of the material. Corrosion which considerably reduces the fatigue strength is started at the surface of a part. Mechanical treatment destroys the integrity of crystalline grains of the material and leaves tool marks on the surface in the form of scratches. Therefore, a number of measures are aimed at making the surface layer of the material harder and improving the fatigue strength of a part in this way. This is particularly efficient in the presence of stress raisers. The hardening is achieved by thermal, thermochemical and special mechanical treatment of the surface producing hardening and plastic deformation in the surface layers of the material. By this means residual compressive stresses are set up in the surface layers of the material,

which are balanced by residual tensile stresses in the interior layers. Under repeated loading, the dangerous stresses are tensile ones. The tensile stresses in the surface layers produced by the load are superimposed on the residual compressive stresses and in consequence the resultant tensile stress is reduced and the part strengthened. Figure 215 shows schematic stress diagrams for a shaft subjected to bending: curve a = diagram of residual stresses produced by hardening of the surface layer during manufacturing; line b = stress diagram

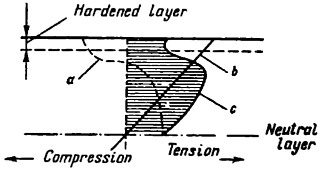


Fig. 215

due to the bending of the shaft; curve c = resultant stress diagram. As is seen from the figure, the tensile stress due to bending is considerably reduced at the surface of the shaft. The increase of compressive stresses in the opposite fibres of the shaft is not dangerous as far as fatigue failure is concerned.

The hardening of the surface layers during manufacturing includes: induction surface hardening, thermochemical surface treatment by nitriding or carburizing, mechanical surface hardening by rolling with hardened steel balls, surface finish by shot peening, etc.

The choice of manufacturing process is governed by the type of material, requirements on surface finish, operating conditions of the part, economics of design and other considerations. Measures resulting in surface hardening are extensively used in mechanical engineering; they increase the fatigue strength of such parts as shafts, axles, springs, bolts, gears, etc. by a score or two per cent.

More detailed information on the application of these measures and the effect they produce may be found in the special literature.

109. Check Questions

State examples of static and dynamic action of loads.

How is the stress determined in a thin ring rotating about a centroidal axis perpendicular to its plane?

What is the relation between the stresses produced by a suddenly applied tensile force and a statically applied force of the same magnitude?

How is the impact factor calculated for a beam under a transverse impact load?

How are impact tests performed?

What is the impact strength or the modulus of toughness of materials?

How are materials tested for fatigue (endurance)?

Define the mean stress and the stress amplitude of a cycle of varying stresses.

What stress cycle is completely reversed?

What is the endurance limit of a material?

What empirical approximate relations exist for predicting the endurance limit of steels for the completely reversed stress cycle from the ultimate strength of the material?

How is the endurance limit of the material influenced by the overall dimensions and surface finish of a part?

What is the effective stress concentration factor?

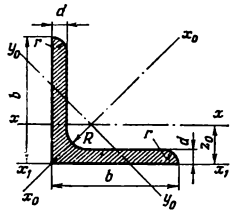
How is the factor of safety determined in the case of completely reversed stresses?

How is the approximate fatigue strength diagram constructed?

How is the factor of safety determined for a part from the approximate fatigue strength diagram?

How is the factor of safety determined in the case of combined varying stresses?

APPENDICES



Rolled Steel Equal Angles
GOST 8509-57

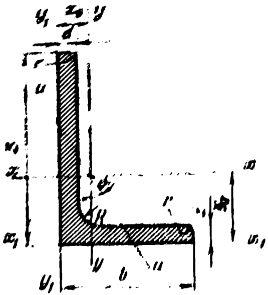
No. of section	Dimensions				Area of section	Weight per metre	Data for axes							
							$x-x$		x_0-x_0		y_0-y_0		x_1-x_1	
	b	d	R	r			I_x	i_x	I_{x_0} max.	i_{x_0} max.	I_{y_0} min.	i_{y_0} min.	I_{x_1}	z_0
	mm				cm^2	kgf	cm^4	cm	cm^4	cm	cm^4	cm	cm^4	cm
2	20	3	3.5	1.2	1.13	0.89	0.40	0.59	0.63	0.75	0.17	0.39	0.81	0.60
		4			1.46	1.15	0.50	0.58	0.78	0.73	0.22	0.38	1.09	0.64
2.5	25	3	3.5	1.2	1.43	1.12	0.81	0.75	1.29	0.95	0.34	0.49	1.57	0.73
		4			1.86	1.46	1.03	0.74	1.62	0.93	0.44	0.48	2.11	0.76
2.8	28	3	4	1.3	1.62	1.27	1.16	0.85	1.84	1.07	0.48	0.55	2.20	0.80
3.2	32	3	4.5	1.5	1.86	1.46	1.77	0.97	2.8	1.23	0.74	0.63	3.26	0.89
		4			2.43	1.91	2.26	0.96	3.58	1.21	0.94	0.62	4.39	0.94
3.6	36	3	4.5	1.5	2.10	1.65	2.56	1.10	4.06	1.39	1.06	0.71	4.64	0.99
		4			2.75	2.16	3.29	1.09	5.21	1.38	1.36	0.70	6.24	1.04

4	40	3	5	1.7	2.35	1.85	3.55	1.23	5.63	1.55	1.47	0.79	6.35	1.09
		4			3.08	2.42	4.58	1.22	7.26	1.53	1.90	0.78	8.53	1.13
4.5	45	3	5	1.7	2.65	2.08	5.13	1.39	8.13	1.75	2.12	0.89	9.04	1.21
		4			3.48	2.73	6.63	1.38	10.5	1.74	2.74	0.89	12.1	1.26
		5			4.29	3.37	8.03	1.37	12.7	1.72	3.33	0.88	15.3	1.30
5.0	50	3	5.5	1.8	2.96	2.32	7.11	1.55	11.3	1.95	2.95	1.00	12.4	1.33
		4			3.89	3.05	9.21	1.54	14.6	1.94	3.80	0.99	16.6	1.38
		5			4.80	3.77	11.2	1.53	17.8	1.92	4.63	0.98	20.9	1.42
5.6	56	3.5	6	2	3.86	3.03	11.6	1.73	18.4	2.18	4.80	1.12	20.3	1.50
		4			4.38	3.44	13.1	1.73	20.8	2.18	5.41	1.11	23.3	1.52
		5			5.41	4.25	16.0	1.72	25.4	2.16	6.59	1.10	29.2	1.57
6.3	63	4	7	2.3	4.96	3.90	18.9	1.95	29.9	2.45	7.81	1.25	33.1	1.69
		5			6.13	4.81	23.1	1.94	36.6	2.44	9.52	1.25	41.5	1.74
		6			7.28	5.72	27.1	1.93	42.9	2.43	11.2	1.24	50.0	1.78
7	70	4.5	8	2.7	6.2	4.87	29.0	2.16	46.0	2.72	12.0	1.39	51.0	1.88
		5			6.86	5.38	31.9	2.16	50.7	2.72	13.2	1.39	56.7	1.90
		6			8.15	6.39	37.6	2.15	59.6	2.71	15.5	1.38	68.4	1.94
		7			9.42	7.39	43.0	2.14	68.2	2.69	17.3	1.37	80.1	1.99
		8			10.7	8.37	48.2	2.13	76.4	2.68	20.0	1.37	91.9	2.02
7.5	75	5	9	3	7.39	5.8	39.5	2.31	62.6	2.91	16.4	1.49	69.6	2.02
		6			8.78	6.89	46.6	2.30	73.9	2.90	19.3	1.48	83.9	2.06
		7			10.1	7.96	53.3	2.29	84.6	2.89	22.1	1.48	98.3	2.10
		8			11.5	9.02	59.8	2.28	94.9	2.87	24.8	1.47	113	2.15
		9			12.8	10.1	66.1	2.27	105	2.86	27.5	1.46	127	2.18

Continued

No. of section	b	d	R	r	Area of section	Weight per metre	I_x	i_x	I_{x_0} max.	i_{x_0} max.	I_{y_0} min.	i_{y_0} min.	I_{x_1}	z_0
	Dimensions						Data for axes: $x-x$		x_0-x_0		y_0-y_0		x_1-x_1	
	mm				cm²	kgf	cm⁴	cm	cm⁴	cm	cm⁴	cm	cm⁴	cm
8	80	5.5	9	3	8.63	6.78	52.7	2.47	83.6	3.11	21.8	1.59	93.2	2.17
		6			9.38	7.36	57.0	2.47	90.4	3.11	23.5	1.58	102	2.19
		7			10.8	8.51	65.3	2.45	104	3.09	27.0	1.58	119	2.23
		8			12.3	9.65	73.4	2.44	116	3.08	30.3	1.57	137	2.27
9	90	6	10	3.3	10.6	8.33	82.1	2.78	130	3.50	34.0	1.79	145	2.43
		7			12.3	9.64	94.3	2.77	150	3.49	38.9	1.78	169	2.47
		8			13.9	10.9	106	2.76	168	3.48	43.8	1.77	194	2.51
		9			15.6	12.2	118	2.75	186	3.46	48.6	1.77	219	2.55
10	100	6.5	12	4	12.8	10.1	122	3.09	193	3.88	50.7	1.99	214	2.68
		7.0			13.8	10.8	131	3.08	207	3.88	54.2	1.98	231	2.71
		8			15.6	12.2	147	3.07	233	3.87	60.9	1.98	265	2.75
		10			19.2	15.1	179	3.05	284	3.84	74.1	1.96	333	2.83
		12			22.8	17.9	209	3.03	331	3.81	86.9	1.95	402	2.91
		14			26.3	20.6	237	3.00	375	3.78	99.3	1.94	472	2.99
		16			29.7	23.3	264	2.98	416	3.74	112	1.94	542	3.06
11	110	7	12	4	15.2	11.9	176	3.40	279	4.29	72.7	2.19	308	2.96
		8			17.2	13.5	198	3.39	315	4.28	81.8	2.18	353	3.00
		8			19.7	15.5	294	3.87	467	4.87	122	2.49	516	3.36
		9			22.0	17.3	327	3.86	520	4.86	135	2.48	582	3.40

12.5	125	10	14	4.6	24.3	19.1	360	3.85	571	4.84	149	2.47	649	3.45
		12			28.9	22.7	422	3.82	670	4.82	174	2.46	782	3.53
		14			33.4	26.2	482	3.80	764	4.78	200	2.45	916	3.61
		16			37.8	29.6	539	3.78	853	4.75	224	2.44	1051	3.68
14	140	9	14	4.6	24.7	19.4	466	4.34	739	5.47	192	2.79	818	3.78
		10			27.3	21.5	512	4.33	814	5.46	211	2.78	911	3.82
		12			32.5	25.5	602	4.31	957	5.43	248	2.76	1,097	3.90
16	160	10	16	5.3	31.4	24.7	774	4.96	1,229	6.25	319	3.19	1,356	4.30
		11			34.4	27.0	844	4.95	1,341	6.24	348	3.18	1,494	4.35
		12			37.4	29.4	913	4.94	1,450	6.23	376	3.17	1,633	4.39
		14			43.3	34.0	1,046	4.92	1,662	6.20	431	3.16	1,911	4.47
		16			49.1	38.5	1,175	4.89	1,866	6.17	465	3.14	2,191	4.55
		18			54.8	43.0	1,299	4.87	2,061	6.13	537	3.13	2,472	4.63
		20			60.4	47.4	1,419	4.85	2,248	6.10	589	3.12	2,756	4.70
18	180	11	16	5.3	38.8	30.5	1,216	5.60	1,933	7.06	500	3.59	2,128	4.85
		12			42.2	33.1	1,317	5.59	2,093	7.04	540	3.58	2,324	4.89
20	200	12	18	6	47.1	37.0	1,823	6.22	2,896	7.84	749	3.99	3,182	5.37
		13			50.9	39.9	1,961	6.21	3,116	7.83	805	3.98	3,452	5.42
		14			54.6	42.8	2,097	6.20	3,333	7.81	861	3.97	3,722	5.46
		16			62.0	48.7	2,363	6.17	3,755	7.78	970	3.96	4,264	5.54
		20			76.5	60.1	2,871	6.12	4,560	7.72	1,182	3.93	5,355	5.70
		25			94.3	74.0	3,466	6.06	5,494	7.63	1,438	3.91	6,733	5.89
		30			111.5	87.6	4,020	6.00	6,351	7.55	1,688	3.89	8,130	6.07
22	220	14	21	7	60.4	47.4	2,814	6.83	4,470	8.60	1,159	4.38	4,941	5.93
		16			68.6	53.8	3,175	6.81	5,045	8.58	1,306	4.36	5,661	6.02
25	250	16	24	8	78.4	61.5	4,717	7.76	7,492	9.78	1,942	4.98	8,286	6.75
		18			87.7	68.9	5,247	7.73	8,337	9.75	2,158	4.96	9,342	6.83
		20			97.0	76.1	5,765	7.71	9,160	9.72	2,370	4.94	10,401	6.91
		22			106.1	83.3	6,270	7.69	9,961	9.69	2,579	4.93	11,464	7.00
		25			119.7	94.0	7,006	7.65	11,125	9.64	2,887	4.91	13,064	7.11
		28			133.1	104.5	7,717	7.61	12,244	9.59	3,190	4.89	14,674	7.23
		30			142.0	111.4	8,117	7.59	12,965	9.56	3,389	4.89	15,753	7.31



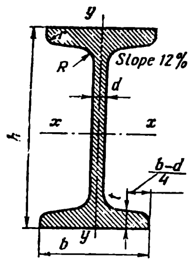
No. of section	B	b	d	R	r	Area of section	Weight per metre	I_x	i_x	I_y	i_y	I_{x_1}	Distance from centre y_0	I_{y_1}	Distance from centre x_0	$I_{u\,min}$	$i_{u\,min}$	Slope of axis
	mm					cm²	kgf	cm⁴	cm	cm⁴	cm	cm⁴	cm	cm⁴	cm	cm⁴	cm	tan α
2.5/1.6	25	16	3	3.5	1.2	1.16	0.91	0.70	0.78	0.22	0.44	1.56	0.86	0.43	0.42	0.13	0.34	0.392
3.2/2	32	20	3	3.5	1.2	1.49	1.17	1.52	1.01	0.46	0.55	3.26	1.08	0.82	0.49	0.28	0.43	0.382
			4			1.94	1.52	1.93	1.00	0.57	0.54	4.38	1.12	1.12	0.53	0.35	0.43	0.374
4/2.5	40	25	3	4.0	1.3	1.89	1.48	3.06	1.27	0.93	0.70	6.37	1.32	1.58	0.59	0.56	0.54	0.385
			4			2.47	1.94	3.93	1.26	1.18	0.69	8.53	1.37	2.15	0.63	0.71	0.54	0.381

4.5/2.8	45	28	3	5	1.7	2.14	1.68	4.41	1.43	1.32	0.79	9.02	1.47	2.20	0.64	0.79	0.61	0.382
			4			2.80	2.20	5.68	1.42	1.69	0.78	12.1	1.51	2.98	0.68	1.02	0.60	0.379
5.3/2	50	32	3	5.5	1.8	2.42	1.90	6.17	1.60	1.99	0.91	12.4	1.60	3.26	0.72	1.18	0.70	0.403
			4			3.17	2.49	7.98	1.59	2.56	0.90	16.6	1.65	4.42	0.76	1.52	0.69	0.401
5.6/3.6	56	36	3.5	6.0	2.0	3.16	2.48	10.1	1.79	3.30	1.02	20.3	1.80	5.43	0.82	1.95	0.79	0.407
			4			3.58	2.81	11.4	1.78	3.70	1.02	23.2	1.82	6.25	0.84	2.19	0.78	0.406
			5			4.41	3.46	13.8	1.77	4.48	1.01	29.2	1.86	7.91	0.88	2.66	0.78	0.404
6.3/4.0	63	40	4	7.0	2.3	4.04	3.17	16.3	2.01	5.16	1.13	33.0	2.03	8.51	0.91	3.07	0.87	0.397
			5			4.98	3.91	19.9	2.00	6.26	1.12	41.4	2.08	10.8	0.95	3.72	0.86	0.396
			6			5.90	4.63	23.3	1.99	7.28	1.11	49.9	2.12	13.1	0.99	4.36	0.86	0.393
			8			7.68	6.03	29.6	1.96	9.15	1.09	66.9	2.20	17.9	1.07	5.58	0.85	0.386
7/4.5	70	45	4.5	7.5	2.5	5.07	3.98	25.3	2.23	8.25	1.28	51	2.25	13.6	1.03	4.88	0.98	0.407
			5			5.59	4.39	27.8	2.23	9.05	1.27	56.7	2.28	15.2	1.05	5.34	0.98	0.406
7.5/5	75	50	5	8	2.7	6.11	4.79	34.8	2.39	12.5	1.43	69.7	2.39	20.8	1.17	7.24	1.09	0.436
			6			7.25	5.69	40.9	2.38	14.6	1.43	83.9	2.44	25.2	1.21	8.48	1.08	0.435
			8			9.47	7.43	52.4	2.35	18.5	1.40	112	2.52	34.2	1.29	10.9	1.07	0.430

Continued

No. of section	Dimensions					Area of section	Weight per metre	Data for axes										
								$x-x$		$y-y$		x_1-x_1		y_1-y_1		$u-u$		
	B	b	d	R	r			I_x	i_x	I_y	i_y	I_{x_1}	Distance from centroid, y_0	I_{y_1} min.	Distance from centroid, x_0	I_u min.	i_u min.	Slope of axis
	mm					cm²	kgf	cm⁴	cm	cm⁴	cm	cm⁴	cm	cm⁴	cm	cm⁴	cm	tan α
8/5	80	50	5	8	2.7	6.36	4.99	41.6	2.56	12.7	1.41	84.6	2.6	20.8	1.13	7.58	1.09	0.387
			6			7.55	5.92	49.0	2.55	14.8	1.40	102	2.65	25.2	1.17	8.88	1.08	0.386
9/5.6	90	56	5.5	9	3	7.86	6.17	65.3	2.88	19.7	1.58	132	2.92	32.2	1.26	11.8	1.22	0.384
			6			8.54	6.70	70.6	2.88	21.2	1.58	145	2.95	35.2	1.28	12.7	1.22	0.384
			8			11.18	8.77	90.9	2.85	26.1	1.56	194	3.04	47.8	1.36	16.3	1.21	0.380
10/6.3	100	63	6	10	3.3	9.59	7.53	98.3	3.2	30.6	1.79	198	3.23	49.9	1.42	18.2	1.38	0.393
			7			11.1	8.70	113	3.19	35.0	1.78	232	3.28	58.7	1.46	20.8	1.37	0.392
			8			12.6	9.87	127	3.18	39.2	1.77	266	3.32	67.6	1.50	23.4	1.36	0.391
			10			15.5	12.1	154	3.15	47.1	1.75	333	3.40	85.8	1.58	28.3	1.35	0.387
11/7	110	70	6.5	10	3.3	11.4	8.98	142	3.53	45.6	2	286	3.55	74.3	1.58	26.9	1.53	0.402
			7			12.3	9.64	152	3.52	48.7	1.99	309	3.57	80.3	1.60	28.8	1.53	0.402
			8			13.9	10.9	172	3.51	54.6	1.98	353	3.61	92.3	1.64	32.3	1.52	0.400

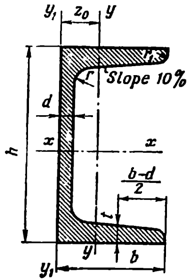
12.5/8	125	80	7	11	3.7	14.1	11	227	4.01	73.7	2.29	452	4.01	119	1.8	43.4	1.76	0.407
			8			16	12.5	256	4	83.0	2.28	518	4.05	137	1.84	48.8	1.75	0.406
			10			19.7	15.5	312	3.98	100	2.26	649	4.14	173	1.92	59.3	1.74	0.404
			12			23.4	18.3	365	3.95	117	2.24	781	4.22	210	2	69.5	1.72	0.400
14/9	140	90	8	12	4	18	14.1	364	4.49	120	2.58	727	4.49	194	2.03	70.3	1.98	0.411
			10			22.2	17.5	444	4.47	146	2.56	911	4.58	245	2.12	85.5	1.96	0.409
16/10	160	100	9	13	4.3	22.9	18	606	5.15	186	2.85	1,221	5.19	300	2.23	110	2.2	0.391
			10			25.3	19.8	667	5.13	204	2.84	1,359	5.23	335	2.28	121	2.19	0.390
			12			30	23.6	784	5.11	239	2.82	1,634	5.32	405	2.36	142	2.18	0.388
			14			34.7	27.3	897	5.08	272	2.8	1,910	5.40	477	2.43	162	2.16	0.385
18/11	180	110	10	14	4.7	28.3	22.2	952	5.8	276	3.12	1,933	5.88	444	2.44	165	2.42	0.375
			12			33.7	26.4	1,123	5.77	324	3.1	2,324	5.97	537	2.52	194	2.40	0.374
20/12.5	200	125	11	14	4.7	34.9	27.4	1,449	6.45	446	3.58	2,920	6.5	718	2.79	264	2.75	0.392
			12			37.9	29.7	1,568	6.43	482	3.57	3,189	6.54	786	2.83	285	2.74	0.392
			14			43.9	34.4	1,801	6.41	551	3.54	3,726	6.62	922	2.91	327	2.73	0.390
			16			49.8	39.1	2,026	6.38	617	3.52	4,264	6.71	1,061	2.99	367	2.72	0.388
25/16	250	160	12	18	6	48.3	37.9	3,147	8.07	1,032	4.62	6,212	7.97	1,634	3.53	604	3.54	0.410
			16			63.6	49.9	4,091	8.02	1,333	4.58	8,308	8.14	2,200	3.69	781	3.50	0.408
			18			71.1	55.8	4,545	7.99	1,475	4.56	9,358	8.23	2,487	3.77	866	3.49	0.407
			20			78.5	61.7	4,987	7.97	1,613	4.53	10,410	8.31	2,776	3.85	949	3.48	0.405



Rolled Steel I Beams
GOST 8239-56

No. of section	Weight per metre, kgf	Dimensions						Area of section, cm^2	Data for axes						
									x—x				y—y		
		h	b	d	t	R	r		I_x	Z_x	i_x	S_x	I_y	Z_y	i_y
		mm							cm^4	cm^3	cm	cm^3	cm^4	cm^3	cm
10	9.46	100	55	4.5	7.2	7	2.5	12.0	198	39.7	4.06	23.0	17.9	6.49	1.22
12	11.5	120	64	4.8	7.3	7.5	3	14.7	350	58.4	4.88	33.7	27.9	8.72	1.38
14	13.7	140	73	4.9	7.5	8	3	17.4	572	81.7	5.73	46.8	41.9	11.5	1.55
16	15.9	160	81	5.0	7.8	8.5	3.5	20.2	873	109	6.57	62.3	58.6	14.5	1.70
18	18.4	180	90	5.1	8.1	9	3.5	23.4	1,290	143	7.42	81.4	82.6	18.4	1.88
18a	19.9	180	100	5.1	8.3	9	3.5	25.4	1,430	159	7.51	89.8	114	22.8	2.12
20	21.0	200	100	5.2	8.4	9.5	4	26.8	1,840	184	8.28	104	115	23.1	2.07

20a	22.7	200	110	5.2	8.6	9.5	4	28.9	2,030	203	8.37	114	155	28.2	2.32
22	24.0	220	110	5.4	8.7	10	4	30.6	2,550	232	9.13	131	157	28.6	2.27
22a	25.8	220	120	5.4	8.9	10	4	32.8	2,790	254	9.22	143	206	34.3	2.50
24	27.3	240	115	5.6	9.5	10.5	4	34.8	3,460	289	9.97	163	198	34.5	2.37
24a	29.4	240	125	5.6	9.8	10.5	4	37.5	3,800	317	10.1	178	260	41.6	2.63
27	31.5	270	125	6.0	9.8	11	4.5	40.2	5,010	371	11.2	210	260	41.5	2.54
27a	33.9	270	135	6.0	10.2	11	4.5	43.2	5,500	407	11.3	229	337	50.0	2.80
30	36.5	300	135	6.5	10.2	12	5	46.5	7,080	472	12.3	268	337	49.9	2.69
30a	39.2	300	145	6.5	10.7	12	5	49.9	7,780	518	12.5	292	436	60.1	2.95
33	42.2	330	140	7.0	11.2	13	5	53.8	9,840	597	13.5	339	419	59.9	2.79
36	48.6	360	145	7.5	12.3	14	6	61.9	13,380	743	14.7	423	516	71.1	2.89
40	56.1	400	155	8.0	13.0	15	6	71.4	18,930	947	16.3	540	666	85.9	3.05
45	65.2	450	160	8.6	14.2	16	7	83.0	27,450	1,220	18.2	699	807	101	3.12
50	76.8	500	170	9.5	15.2	17	7	97.8	39,290	1,570	20.0	905	1,040	122	3.26
55	89.8	550	180	10.3	16.5	18	7	114	55,150	2,000	22.0	1,150	1,350	150	3.44
60	104	600	190	11.1	17.8	20	8	132	75,450	2,510	23.9	1,450	1,720	181	3.60
65	120	650	200	12.0	19.2	22	9	153	101,400	3,120	25.8	1,800	2,170	217	3.77
70	138	700	210	13.0	20.8	24	10	176	134,600	3,840	27.7	2,230	2,730	260	3.94
70a	158	700	210	15.0	24.0	24	10	202	152,700	4,360	27.5	2,550	3,240	309	4.01
70b	184	700	210	17.5	28.2	24	10	234	175,370	5,010	27.4	2,940	3,910	373	4.09



Rolled Steel Channels
GOST 8240-56

No. of section	Weight per metre, kgf	Dimensions						Area of section, cm^2	Data for axes							
									$x-x$				$y-y$			z_0
		h	b	d	t	r	r_1		I_x	Z_x	i_x	S_x	I_y	Z_y	i_y	
		mm							cm^4	cm^3	cm	cm^3	cm^4	cm^3	cm	cm
5	4.84	50	32	4.4	7.0	6.0	2.5	6.16	22.8	9.1	1.92	5.59	5.61	2.75	0.954	1.16
6.5	5.90	65	36	4.4	7.2	6.0	2.5	7.51	48.6	15.0	2.54	9.00	8.70	3.68	1.08	1.24
8	7.05	80	40	4.5	7.4	6.5	2.5	8.98	89.4	22.4	3.16	13.3	12.8	4.75	1.19	1.31
10	8.59	100	46	4.5	7.6	7.0	3.0	10.9	174	34.8	3.99	20.4	20.4	6.46	1.37	1.44
12	10.4	120	52	4.8	7.8	7.5	3.0	13.3	304	50.6	4.78	29.6	31.2	8.52	1.53	1.54
4	12.3	140	58	4.9	8.1	8.0	3.0	15.6	491	70.2	5.60	40.8	45.4	11.0	1.70	1.67

14a	13.3	140	62	4.9	8.7	8.0	3.0	17.0	545	77.8	5.66	45.1	57.5	13.3	1.84	1.87
16	14.2	160	64	5.0	8.4	8.5	3.5	18.1	747	93.4	6.42	54.1	63.3	13.8	1.87	1.80
16a	15.3	160	68	5.0	9.0	8.5	3.5	19.5	823	103	6.49	59.4	78.8	16.4	2.01	2.00
18	16.3	180	70	5.1	8.7	9.0	3.5	20.7	1,090	121	7.24	69.8	86.0	17.0	2.04	1.94
18a	17.4	180	74	5.1	9.3	9.0	3.5	22.2	1,190	132	7.32	76.1	105	20.0	2.18	2.13
20	18.4	200	76	5.2	9.0	9.5	4.0	23.4	1,520	152	8.07	87.8	113	20.5	2.20	2.07
20a	19.8	200	80	5.2	9.7	9.5	4.0	25.2	1,670	167	8.15	95.9	139	24.2	2.35	2.28
22	21.0	220	82	5.4	9.5	10.0	4.0	26.7	2,110	192	8.89	110	151	25.1	2.37	2.21
22a	22.6	220	87	5.4	10.2	10.0	4.0	28.8	2,330	212	8.99	121	187	30.0	2.55	2.46
24	24.0	240	90	5.6	10.0	10.5	4.0	30.6	2,900	242	9.73	139	208	31.6	2.60	2.42
24a	25.8	240	95	5.6	10.7	10.5	4.0	32.9	3,180	265	9.84	151	254	37.2	2.78	2.67
27	27.7	270	95	6.0	10.5	11	4.5	35.2	4,160	308	10.9	178	262	37.3	2.73	2.47
30	31.8	300	100	6.5	11.0	12	5.0	40.5	5,810	387	12.0	224	327	43.6	2.84	2.52
33	36.5	330	105	7.0	11.7	13	5.0	46.5	7,980	484	13.1	281	410	51.8	2.97	2.59
36	41.9	360	110	7.5	12.6	14	6.0	53.4	10,820	601	14.2	350	513	61.7	3.10	2.68
40	48.3	400	115	8.0	13.5	15	6.0	61.5	15,220	761	15.7	444	642	73.4	3.23	2.75

INDEX

Printed in the Union of Soviet Socialist Republics